Effective Ginzburg-Landau model for superfluid ^{4}He

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We propose an effective Ginzburg-Landau energy functional for superfluid helium that is motivated by the ideal Bose gas model. The parameters of this phenomenological model are adjusted to known properties of the bulk system. The motivation and the field of applications of this model are quite similar to that of the Ψ theory by Ginzburg and Sobyanin. The correlation length, the depression of the transition temperature in helium films and the density profile at a boundary are calculated.

I. INTRODUCTION

In the Ginzburg-Landau (GL) model for liquid ^{4}He the superfluid density ϱ_s is identified with the expectation value of the square of the complex order parameter field $\Psi(\mathbf{r})$,

$$\varrho_s = m \left\langle |\Psi|^2 \right\rangle \ . \qquad (1)$$

Here m is the mass of a helium atom. In the vicinity of the λ-transition, the energy functional is expanded into powers of the field Ψ and its derivatives and of the relative temperature $t = T/T_\lambda - 1$. In its simplest form the GL energy functional takes into account the terms with $|\nabla\Psi|^2$, $t \cdot |\Psi|^2$ and $|\Psi|^4$.

The GL model for superfluid helium has been introduced in 1958 by Ginzburg and Pitaevskiĭ[1]. It leads to critical properties (like $\varrho_s \propto |t|$) that do not agree with the experiment. This failure is due to the fact that in this model the fluctuations diverge (relative to the mean value) when the transition point is approached.

The theoretical remedy for this break-down of the GL model is well-known: The GL ansatz may be considered as a sensible approximation for finite volumes of the sample only. Using scale transformations to larger and larger volumes renormalizes the coefficients in the energy functional. The renormalization group theory (RGT) leads to nontrivial critical exponents that are no longer in variance with experiment. RGT is considered as the best present-day theory which allows in principle the calculation of all relevant quantities.

For practical reasons it is, however, useful to have a phenomenological GL like model that is in accordance with the bulk properties and in particular with the experimental critical exponents of superfluid helium. Such an effective GL model allows to address a large variety of problems in a simple and straight-forward way.

An effective GL model of this kind has been put forward under the name "Ψ theory" by Ginzburg and Sobyanin[2–4]. The references 2–4 are comprehensive papers addressing the foundation and applications of such an approach. For the historical development of the field we refer to the corresponding discussion and the bibliography in these papers.

The Ψ theory by Ginzburg and Sobyanin (abbreviated by GS in the following) is based on Eq. (1), too. The GS energy functional contains the terms $|\nabla\Psi|^2$, $|t|^{4/3} \cdot |\Psi|^2$, $|t|^{2/3} \cdot |\Psi|^4$ and $|\Psi|^6$. The temperature dependences of the coefficients are chosen by hand such that they yield approximately the right critical exponents (like $\varrho_s \propto |t|^{2/3}$, for example). In so far as the actual critical exponent of ϱ_s deviates from $\nu = 1/3$ this model is an approximation. For the sensibility and usefulness of such an effective GL model and its relation to RGT we refer to the thorough discussion by GS[2–4].

It is generally accepted that there is an intimate connection[5–7] between the Bose-Einstein condensation of the ideal Bose gas (IBG) and the λ transition in liquid helium. In Ref. 8 we have argued that there are good reasons to uphold the IBG critical exponent $\beta = 1/2$ of the condensate density $\varrho_0 = m \langle |\Psi|^2 \rangle \sim |t|$. This implies $\varrho_0 < \varrho_s$ just below the lambda point. Consequently, we have proposed that the superfluid density is composed by the condensate density and a coherently comoving density ϱ_{coh} (corresponding to lowlying noncondensed states):

$$\varrho_s = \varrho_0 + \varrho_{coh} \ , \quad \text{where} \quad \varrho_0 = m \left\langle |\Psi|^2 \right\rangle \ . \qquad (2)$$

The arguments in favor of this composition of ϱ_s and the resulting expression for ϱ_s are reviewed in Sec. II A. The expression for ϱ_s provides excellent fits[8] to the experimental temperature dependence of the superfluid density.

Based on the starting point (2) instead of Eq. (1) we construct in this paper an alternative effective GL model. Because of $\varrho_0 = m \langle |\Psi|^2 \rangle \sim |t|$ we may use the standard form for the Landau part of the energy functional (i. e. the terms $t \cdot |\Psi|^2$ and $|\Psi|^4$). The kinetic energy contribution (the term with $|\nabla\Psi|^2$) gets, however, an additional factor ϱ_s/ϱ_0 due to the coherent comotion. Effectively, this factor $\varrho_s/\varrho_0 \sim |t|^{-1/3}$ damps the critical fluctuation such that our GL ansatz becomes scaling invariant.

From a pragmatic point of view our effective GL model (with the terms $|t|^{-1/3} \cdot |\nabla\Psi|^2$, $t \cdot |\Psi|^2$ and $|\Psi|^4$) is quite comparable to the Ψ theory by GS (with the terms $|\nabla\Psi|^2$, $|t|^{4/3} \cdot |\Psi|^2$, $|t|^{2/3} \cdot |\Psi|^4$ and $|\Psi|^6$). The main difference is that in our model the Landau part is of its standard form (in accordance with the IBG), and that specific reasons are given for the occurrence of a fractional power of $|t|$.

This paper is organized as follows. Section II presents the reasons for the composition (2) of the superfluid density, the actual expression for $\varrho_{\rm s}$, and the parameters obtained by a fit to the experimental temperature dependence of $\varrho_{\rm s}$. Section III displays the energy functional of our effective GL model, and discusses its basic features. Section IV investigates the temperature dependence of the correlation length and the size of the critical fluctuations. The depression of the transition temperature in helium films is calculated in Sec. V. Section VI derives the spatial variation of the superfluid and the condensate density at a boundary.

II. COMPOSITION OF THE SUPERFLUID DENSITY

A. Motivation

The comparison between liquid ^{3}He and ^{4}He on one side and the ideal Fermi gas and the ideal Bose model (IBG) on the other side strongly suggests that there is an intimate relation between the Bose-Einstein condensation (BEC) and the lambda transition in liquid ^{4}He. It is, moreover, inviting to identify Ψ in Eq. (1) with the IBG condensate wave function. This identification explains[6,7] a number of experimental findings of which the most important one[9] is that a superfluid current is irrotational. The IBG *and* this identification are, however, in conflict with the experimental critical exponent $\nu \approx 1/3$ of $\varrho_{\rm s} \sim |t|^{2\nu}$ (as compared to the IBG with $\beta = 1/2$ for $\langle|\Psi|^2\rangle \sim |t|^{2\beta}$).

The standard solution of this conflict appears to be the renormalization group theory (RGT). The RGT yields indeed an explanation of the experimental value of $\nu \approx 1/3$. It does, however, not resolve the conflict with the IBG value $\beta = 1/2$: A renormalization is appropriate for the Landau value $\beta_{\rm L} = 1/2$ but not for the IBG value $\beta = 1/2$. The reason is that the IBG value is obtained by the exact evaluation of a partition sum. This exact evaluation implies a summation over arbitrarily small momenta (or, correspondingly, arbitrarily large distances). Therefore, the reasoning behind the renormalization procedure (analytic Ginzburg-Landau ansatz for *finite* regions or blocks, and subsequent transformation to larger and larger blocks) cannot be applied to the IBG energy functional.

Within the framework of the ideal Bose gas model we have proposed[8] to resolve the conflict $\beta = 1/2$ and $\nu \approx 1/3$ by the assumption that *noncondensed particles move coherently with the condensate*. This means that we no longer identify the condensate with the superfluid fraction; the condensate is only part of the superfluid phase. The coherent comotion is constructed in the following way: The condensate wave function may be written as

$$\Psi(\mathbf{r}) = \varphi_0(\mathbf{r}) \exp[\,{\rm i}\,\Phi(\mathbf{r})]\;, \tag{3}$$

where $\varphi_0(\mathbf{r})$ and $\Phi(\mathbf{r})$ are real functions. The velocity of a potential superfluid flow is given by $\mathbf{v}_{\rm s} = (\hbar/m)\nabla\Phi$. The single particle function $\varphi_{\mathbf{k}}(\mathbf{r})$ of the noncondensed states can be chosen as real functions. If now the lowlying states (up to a coherence limit $k_{\rm coh}$ that is to be determined later) adopt the same phase factor,

$$\varphi_{\mathbf{k}}(\mathbf{r}) \;\to\; \varphi_{\mathbf{k}}(\mathbf{r}) \exp[\,{\rm i}\,\Phi(\mathbf{r})]\;, \qquad (k \le k_{\rm coh}) \tag{4}$$

then the corresponding particles contribute to the superfluid current density. Consequently, the superfluid density is made up by the condensate density ϱ_0 and the density $\varrho_{\rm coh}$ of the coherently comoving particles:

$$\frac{\varrho_s}{\varrho} = \frac{1}{N}\left(\langle n_0\rangle + {\sum_{k\le k_{\rm coh}}}' \langle n_k\rangle\right) = \frac{\varrho_0 + \varrho_{\rm coh}}{\varrho}\;. \tag{5}$$

Here $\langle n_k\rangle$ are average occupation numbers of the (slightly modified) IBG, and the prime at the summation symbol excludes the condensed particles from this sum.

We emphasize that the construction (3) to (5) preserves the basic properties of the superfluid flow (like its irrotational behavior). In section III C we discuss the physical meaning of $k_{\rm coh}$ and present an argument why the lowest single particle levels might adopt the same phase factor as the condensate.

B. Functional form of the superfluid density

We determine the asymptotic form of the expression (5). The IBG expectation values $\langle n_k\rangle$ are given by

$$\langle n_k\rangle = \frac{1}{\exp[(\varepsilon_k-\mu)/k_{\rm B}T]-1} = \frac{1}{\exp(\kappa^2+\tau^2)-1}\;. \tag{6}$$

Here μ is the chemical potential, $\varepsilon_k = \hbar^2k^2/2m$ are the single-particle energies, and $k_{\rm B}$ is Boltzmann's constant. In the last expression we introduced the dimensionless momentum

$$\kappa = \frac{\lambda\,|\mathbf{k}|}{\sqrt{4\pi}}\;, \quad \text{where} \quad \lambda = \frac{2\pi\hbar}{\sqrt{2\pi m k_{\rm B}T}}\;. \tag{7}$$

The chemical potential μ has been expressed by the dimensionless quantity $\tau^2 = -\mu/k_{\rm B}T$ in Eq. (6). We expand this τ into powers of the relative temperature $t = T/T_\lambda - 1$:

$$\tau(t) = \sqrt{\frac{-\mu}{k_{\rm B}T}} = \begin{cases} a\,t + b\,t^2 + \dots & (t>0) \\ a'\,|t| + b'\,t^2 + \dots & (t<0) \end{cases}\;. \tag{8}$$

For $t > 0$ the particle number condition $N = \sum_{\mathbf{k}}\langle n_{\mathbf{k}}\rangle$ determines the temperature dependence of $\tau(t)$ and in particular the coefficients a, b,..., for example $a = 3\,\zeta(3/2)/(4\pi^{1/2})$. (Here $\zeta(x) = \sum_1^\infty n^{-x}$ denotes Riemann's zeta function). For $t < 0$ the IBG yields $\tau = 0$.

As a modification of the IBG we admit nonvanishing coefficients a', b',.... By $a' \neq 0$ we introduce a phenomenological gap between the condensate level and the noncondensed levels. This modification does not affect the BEC as the most important feature of the IBG. It is necessary in order to reproduce the experimental temperature dependence of the superfluid density by Eq. (5). In view of the successful roton picture it is not surprising that a gap is required for a quantitative description.

For $t < 0$ the particle number condition $N = \langle n_0 \rangle + \sum'_{\mathbf{k}} \langle n_{\mathbf{k}} \rangle$ leads to

$$\frac{\varrho_0}{\varrho} = \frac{\langle n_0 \rangle}{N} = f\,|t| + \mathcal{O}(t^2) \ , \tag{9}$$

where

$$f = \frac{3}{2} + \frac{2\sqrt{\pi}\,a'}{\zeta(3/2)} \ . \tag{10}$$

An evaluation of the coherently comoving density in (5) yields

$$\frac{\varrho_{\rm coh}}{\varrho} = \frac{1}{N} \sum_{k \le k_{\rm coh}}{}' \langle n_k \rangle = \frac{4\,\kappa_{\rm coh}}{\sqrt{\pi}\,\zeta(3/2)} + \text{higher order} \ . \tag{11}$$

The leading experimental behavior is approximately

$$\frac{\varrho_{\rm s}}{\varrho} \approx \frac{\varrho_{\rm coh}}{\varrho} \approx a_0\,|t|^{2/3} \ . \tag{12}$$

Deviations from the exponent 2/3 are neglected in our phenomenological model. Equation (12) corresponds to

$$\kappa_{\rm coh} = \kappa_0\,|t|^{2/3} \ , \tag{13}$$

where

$$a_0 = \frac{4\,\kappa_0}{\sqrt{\pi}\,\zeta(3/2)} \ . \tag{14}$$

C. Results

An excellent agreement with the experimental temperature dependence of the superfluid density is obtained[8] if higher order terms in Eqs. (9) and (11) are included. The resulting fit formula contains four parameters as do comparable standard fits.

This fit defines the temperature dependence of the contributions $\varrho_{\rm coh}$ and ϱ_0 to the superfluid density. The result is shown in Fig. 1 that has been taken from Ref. 8. For $|t| < 0.01$ one finds $\varrho_{\rm s} \approx \varrho_{\rm coh}$ (because of $\varrho_{\rm coh} \sim |t|^{2/3}$ and $\varrho_0 \sim |t|$). Due to the gap between the condensed and the noncondensed particles, the contribution $\varrho_{\rm coh}$ is substantial only in a narrow region ($|t| < 0.2$). For $|t| > 0.2$ one obtains $\varrho_{\rm s} \approx \varrho_0$.

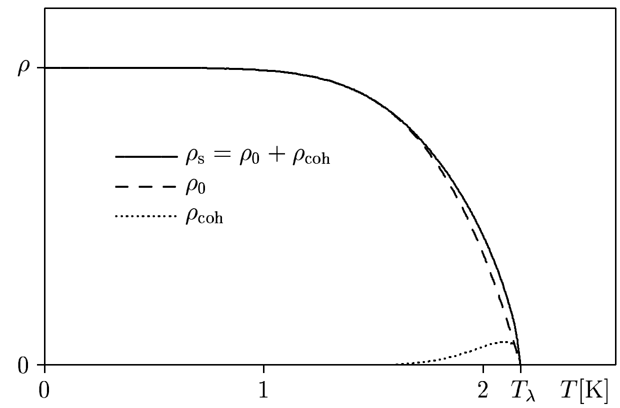

FIG. 1. Decomposition of the superfluid density $\rho_{\rm s}$ into the model condensate density ρ_0 and the coherently comoving density $\rho_{\rm coh}$ as a function of the temperature T. For $T \to T_\lambda$ the comoving density $\rho_{\rm coh} \propto |t|^{2/3}$ is the dominant contribution to the superfluid density.

The model condensate density ϱ_0 may be multiplied by a depletion factor (roughly 0.1) in order to become comparable to the measurable condensate density. The temperature dependence of the resulting condensate density compares well with experimental results by Snow et al.[10].

In this paper we need only the coefficients of the leading terms in Eqs. (9) and (12). The fit of the theoretical expression to the experimental data yields the following parameter values[8]

$$a' = 3.03 \ , \quad \kappa_0 = 2.70 \ . \tag{15}$$

According to Eqs. (10) and (14) these parameters are equivalent to

$$f = 5.60 \ , \quad a_0 = 2.33 \ . \tag{16}$$

These two parameters are used as an input for our energy functional.

III. EFFECTIVE GINZBURG-LANDAU FUNCTIONAL

A. Definition

The equilibrium value of Gibbs free energy G depends on the temperature T, the pressure P and the total particle number N. In a GL like model, the thermodynamic potential G depends, moreover, on the order parameter field Ψ. The minimum of G with respect to Ψ determines its expectation value. The equilibrium value of the potential is then $G(T, P, N) = \langle G(\Psi) \rangle$.

The number of particles is conserved in the considered applications. The pressure dependence will not be discussed explicitly; it enters via $T_\lambda(P)$ and via the pressure dependence of the parameters (16) that may be determined by the fit to the experimental superfluid density $\varrho_{\rm s}(T,P)$. The following considerations concentrate on the t and Ψ dependences.

Equations (2) and (9) yield the bulk equilibrium value

$$v\left\langle|\Psi(\text{bulk})|^2\right\rangle=\frac{\langle n_0\rangle}{N}=f\,|t|+\ldots\;. \tag{17}$$

Here $v=V/N$ is the volume per particle. The order parameter field $\Psi(\mathbf{r})$ will, in general, depend on the coordinates. The thermodynamic potential is, therefore, written as an energy functional.

We split the thermodynamic potential into several parts:

$$G=G_0+G_{\rm GL}=G_0+G_{\rm fluct}+G_{\rm L}\;. \tag{18}$$

The contribution that is not directly related to the order parameter field is denoted by G_0. Following GS[2,3] we assume that the approximately logarithmic singularity of the specific heat is due to this term. Our investigation is restricted to the Ginzburg-Landau (GL) contribution $G_{\rm GL}$ that is due to the order parameter field. This contribution may be further split into the Landau part $G_{\rm L}$ and a part $G_{\rm fluct}$ that contains derivatives of the field Ψ.

Near the transition point the energy functional is expanded into powers of the order parameter field:

$$G_{\rm GL}=\int d^3r\,\left(A\,|\nabla\Psi|^2+B\,|\Psi|^2+C\,|\Psi|^4+\ldots\right)\;. \tag{19}$$

First we consider the terms without derivatives, i.e. the Landau part $G_{\rm L}$. The most simple standard Landau ansatz ($B=b\,t$, $C=$ const. and no higher order terms) leads to the IBG like behavior (17). The coefficient in Eq. (17) is reproduced if we set $G_{\rm L}\propto t\,|\Psi|^2/f+v|\Psi|^4/(2f^2)$. For the determination of the prefactor we follow GS[2,3] who argue that $G_{\rm L}$ should account for the jump of the specific heat at the lambda point. This leads to

$$\frac{G_{\rm L}}{k_{\rm B}T_\lambda}=\int d^3r\,\frac{\Delta c_P}{k_{\rm B}}\left(t\,\frac{|\Psi|^2}{f}+\frac{v\,|\Psi|^4}{2f^2}\right)\;. \tag{20}$$

where Δc_P is the jump of the specific heat per particle. The bulk expectation value of the r.h.s. for $t<0$ is $-V(\Delta c_P/k_{\rm B})|t|^2/(2v)$ yielding $\langle G_{\rm L}/N\rangle=-\Delta c_{\rm P}T_\lambda|t|^2/2$. For $t>0$ we have $\langle G_{\rm L}/N\rangle=0$. The second temperature derivative of these expectation values reproduces the jump of the specific heat.

The contribution $G_{\rm fluct}$ may be written as

$$\frac{G_{\rm fluct}}{k_{\rm B}T_\lambda}=\frac{1}{k_{\rm B}T_\lambda}\int d^3r\,\frac{\hbar^2}{2m}\,R\,|\nabla\Psi|^2\;. \tag{21}$$

In a standard GL ansatz the factor R equals 1. According to Eqs. (3) – (5) the density $\varrho_{\rm coh}$ moves coherently with the condensate density ϱ_0. The term $|\nabla\Psi|^2$ corresponds to the product of ϱ_0 times a squared velocity. This has to be replaced by $\varrho_{\rm s}$ times the squared velocity. This implies

$$R(t)=\frac{\varrho_{\rm s}(\text{bulk})}{\varrho_0(\text{bulk})}\simeq\frac{a_0\,|t|^{2/3}}{f\,|t|}\,. \tag{22}$$

For the last expression we used Eqs. (9) and (12). By $R\sim|t|^{-1/3}$ we admit a fractional power of the relative temperature in the coefficients (similarly as in the Ψ theory by GS).

From Eqs. (20), (21) and (22) we obtain the central expression for the energy functional:

$$\frac{G_{\rm GL}}{k_{\rm B}T_\lambda}=\int d^3r\left[\frac{a_0\,\lambda_{\rm c}^2\,|\nabla\Psi|^2}{4\pi f\,|t|^{1/3}}+\frac{\Delta c_P}{k_{\rm B}}\left(\frac{t\,|\Psi|^2}{f}+\frac{|\Psi|^4}{2f^2}\right)\right]. \tag{23}$$

Here $\lambda_{\rm c}$ denotes the thermal wavelength at the critical temperature:

$$\frac{\lambda_{\rm c}}{\sqrt{4\pi}}=\frac{\hbar}{\sqrt{2mk_{\rm B}T_\lambda}}\approx 1.67\,\text{Å}\;. \tag{24}$$

For the numerical value we inserted the mass $m=m(\text{He})$ of a helium atom and $T_\lambda\approx 2.17\,$K. The other coefficients in the energy functional are given by

$$a_0=2.33\;,\quad f=5.60\;,\quad \frac{\Delta c_P}{k_{\rm B}}=2.77\;. \tag{25}$$

The first two parameters are determined by the fit of Eq. (5) to the experimental superfluid density (Sec. II C). The value for Δc_P is taken from Table V (third line) of Ref. 11. To some extent, the parameter values depend the other terms in the corresponding fit formula used.

We use preferably dimensionless quantities: Both sides of Eq. (23) are dimensionless. The integral $\int d^3r\,|\Psi|^2$ is dimensionless and of order N. The product $\lambda_c\nabla$ is dimensionless, too. The occurring numerical factors (a_0, f and $\Delta c_P/k_{\rm B}$) are of order 1.

B. Discussion

1. Comparison to other approaches

We consider the functional

$$G_{\rm GL}=\int d^3r\,\left(A\,|\nabla\Psi|^2+B\,|\Psi|^2+C\,|\Psi|^4+D\,|\Psi|^6\right)\;. \tag{26}$$

and compare the coefficients of our model with that of a standard Ginzburg-Landau ansatz (GL) and of the Ψ theory (GS):

	A	B	C	D
GL[1]	1	t	1	0
GS[2–4]	1	$\|t\|^{4/3}$	$\|t\|^{2/3}$	1
this work	$\|t\|^{-1/3}$	t	1	0

(27)

The entrance 1 stands for a nonvanishing constant. For GS the $|\Psi|^6$ term must be included since it scales with t^2 (as do the terms $|t|^{4/3}|\Psi|^2$ and $|t|^{2/3}|\Psi|^4$). Because of this term the GS functional contains an additional adjustable parameter. In the standard GL as well as in our approach the $|\Psi|^6$ term is a higher order term.

Due to the connection to the IBG, our Landau part (coefficients B, C and D) coincides with the standard GL form. The occurrence of the $|t|^{-1/3}$ dependence in the fluctuation term (coefficient A) is due to the coherent comotion of noncondensed particle with the condensate.

As far as the results are concerned, our approach is more similar to GS than to GL. In particular the correlation length scales with $|t|^{-2/3}$ as in the Ψ theory, and not with $|t|^{-1/2}$ as for GL.

2. *Choice of coefficients*

The prefactor of the Landau part (20) has been chosen such that the expectation value $\langle G_{\rm L}\rangle$ reproduces the jump Δc_P of specific heat at the lambda point. This choice may be criticized for two reasons: (i) The fluctuation part $\langle G_{\rm fluct}\rangle$ leads to a t^2 term for $t<0$, and will contribute to the jump, too. (ii) There might be a contribution to the jump from the part $\langle G_0\rangle$ in Eq. (18). By assuming that the jump is solely due to the Landau part $\langle G_{\rm L}\rangle$ we follow for simplicity GS. The alternative is to replace $\Delta c_P/k_{\rm B}$ by a unknown parameter. Adjusting this parameter later (for example to the correlation length amplitude) would yield a value rather close to $\Delta c_P/k_{\rm B}$. This might be considered as a pragmatic reasoning of the present choice.

The factor $R=\varrho_{\rm s}/\varrho_0$ introduced in Eq. (21) has been motivated by the coherent comotion (Eqs. (3) to (5)) of noncondensed particles. This comotion refers to a potential superfluid flow with $\mathbf{v}_{\rm s}=(\hbar/m)\nabla\Phi(\mathbf{r})$. The motivation for the R-factor has thus been based on a rather specific coordinate dependence of $\Psi(\mathbf{r})$ for $t<0$. It is, however, the spirit of a GL like approach to consider only one functional for the complex $\Psi(\mathbf{r})$ field, i.e. to use the same coefficients for all occurring coordinate dependences, and for $t<0$ and $t>0$. Alternatively, one could introduce different coefficients for the coordinate variation of the amplitude and the phase of the $\Psi(\mathbf{r})$ field; this would result in a more complicated model.

We have used the asymptotic form $R\sim|t|^{-1/3}$ in the functional (23). This is justified only as long as the leading asymptotic expressions $\varrho_{\rm s}\sim|t|^{2/3}$ and $\varrho_0\sim|t|$ are good approximations. This is the case for

$$|t|\lesssim 0.01 \quad \text{(range of applicability)}\,. \tag{28}$$

The fit[8] of Eq. (5) to the experimental superfluid provides the temperature dependence of the ratio $R(t)$ in a wider range. Using this more general function $R(t)$, our approach might be extended into the less asymptotic region, for example to $0.1>|t|>0.01$, where a GL like approach is still meaningful. The emphasis of the present work is, however, on the temperature range (28).

3. *Coordinate dependence of the order parameter*

We discuss the coordinate dependence of the order parameter field and its relation to the condensate and superfluid density. The order parameter field may be written as

$$\Psi(\mathbf{r})=|\Psi(\mathbf{r})|\exp[\,\mathrm{i}\,\Phi(\mathbf{r})]\,. \tag{29}$$

We are thus dealing with two real functions, the amplitude $|\Psi(\mathbf{r})|$ and the phase $\Phi(\mathbf{r})$, respectively. These functions are determined by the field equations that follow from the energy functional (23) and that are supplemented by suitable boundary conditions.

According to Eqs. (3) to (5) all particles contributing to the superfluid density adopt the phase $\Phi(\mathbf{r})$. The common phase factor has, however, no implication on the coordinate dependences of $\varrho_0(\mathbf{r})$ and $\varrho_{\rm coh}(\mathbf{r})$.

The order parameter field is directly connected to condensate density, Eq. (2). This implies

$$\varrho_0(\mathbf{r})=m\left\langle|\Psi(\mathbf{r})|^2\right\rangle\,. \tag{30}$$

It is not a priori clear how to determine the coordinate dependence of $\varrho_{\rm coh}(\mathbf{r})$. Eq. (22) fixes the bulk relation

$$\varrho_{\rm s}(\text{bulk})=m R(t)\left\langle|\Psi(\text{bulk})|^2\right\rangle \tag{31}$$

between the Ψ field and the superfluid density, only. The coordinate dependence of $\varrho_{\rm s}(\mathbf{r})$ versus that of $|\Psi(\mathbf{r})|^2$ will be discussed in Sec. VI for the important case of the behavior at a boundary.

C. Fluctuations of the phase field

A superfluid flow with the velocity $\mathbf{v}_{\rm s}=(\hbar/m)\nabla\Phi$ constitutes a nonequilibrium excitation of the phase field $\Phi(\mathbf{r})$ in Eq. (29). At finite temperatures there will be also statistical excitations of the phase field. In the following we discuss such thermal excitations or fluctuations. This discussion leads to a qualitative argument for the existence and the meaning of the coherence limit $k_{\rm coh}$ in Eq. (4).

We assume an order parameter field of the form

$$\Psi(\mathbf{r})=\sqrt{\frac{n_0}{V}}\,\exp[\,\mathrm{i}\,\Phi(\mathbf{r})]\,. \tag{32}$$

This means that we do not consider spatial variations of $|\Psi(\mathbf{r})|$ in this subsection.

The average momentum of the fluctuations of the phase field is denoted by

$$k_{\rm fluct}=\langle|\nabla\Phi|\rangle\,. \tag{33}$$

This momentum $k_{\rm fluct}$ will be a function of the temperature.

For the single-particle states of the IBG we use real functions $\varphi_{\mathbf{k}}$. We may then consider the possibility of phase fluctuations for a low-lying state with $n_{\mathbf{k}} \gg 1$, i. e. phase fields $\Phi_{\mathbf{k}}(\mathbf{r})$ that are introduced by the replacement

$$\varphi_{\mathbf{k}} \longrightarrow \varphi_{\mathbf{k}} \exp[\,\mathrm{i}\,\Phi_{\mathbf{k}}(\mathbf{r})] \;. \tag{34}$$

Let us first assume that these additional phases vanish, $\Phi_{\mathbf{k}} = 0$. In this case, the average kinetic energy $\hbar^2 k_{\rm fluct}^2/2m$ of a condensed particle would exceed that of a noncondensed particle with $k < k_{\rm fluct}$. The energy sequence of the single-particle states is, however, a prerequisite of the BEC; the condensate must be formed by the particles with the lowest energy. In order to preserve the energy sequence of the low-lying states we require the phase ordering

$$\Phi_{\mathbf{k}}(\mathbf{r}) = \Phi(\mathbf{r}) \quad \text{for} \quad k \le k_{\rm fluct} \;. \tag{35}$$

This argument does not apply to the states with momenta above $k_{\rm fluct}$.

It is plausible to assume the phase ordering Eq. (35) not only for statistical but also for nonstatistical excitations. This means that the coherence limit introduced in Eq. (4) should be identified with the average momentum of the fluctuations:

$$k_{\rm coh} = k_{\rm fluct} \;. \tag{36}$$

In a first approach, the coherence limit $k_{\rm coh}$ in Eq. (5) can be considered as a parameter that has to be determined from the fit to the experimental superfluid density. In contrast to this point of view, we present now theoretical arguments for the leading asymptotic behavior of $k_{\rm coh}$.

The critical exponents of $\varrho_{\rm s}$ and $k_{\rm fluct}$ may be derived from the following assumptions:

1. The Landau part $G_{\rm L}$ of the thermodynamic potential has its well-known form (compatible with the IBG).

2. The fluctuation part $G_{\rm fluct}$ contains the extra factor $\varrho_{\rm s}/\varrho_0$ due to the assumed comotion of noncondensed particles (as compared to a standard GL functional).

3. The fluctuation part scales in the same way as the Landau part. This scaling invariance is the decisive physical assumption.

We write $G_{\rm GL} = \int d^3r\, g_{\rm GL}$ for the thermodynamic potential. The first two assumptions lead to an integrand of the form

$$g_{\rm GL} = g_{\rm fluct} + g_{\rm L} = \frac{\hbar^2}{2m}\,\frac{\varrho_s}{\varrho_0}\,|\nabla\Psi|^2 + r\,t\,|\Psi|^2 + u\,|\Psi|^4 \;. \tag{37}$$

In contrast to Eq. (23), we do not introduce an assumption about the temperature dependence of the ratio ϱ_s/ϱ_0. The present discussion will rather deduce this temperature dependence.

Asymptotically, the Landau part behaves like

$$\langle g_{\rm L}\rangle \propto r\,t\,\varrho_0 + u\,\varrho_0^2 \sim t^2 \;. \tag{38}$$

The third assumption, the scaling invariance, implies that $\langle g_{\rm fluct}\rangle$ must have the same leading t dependence. Because of $\langle|\nabla\Psi|^2\rangle = \varrho_0\, k_{\rm fluct}^2$ we have $\langle g_{\rm fluct}\rangle \propto \varrho_{\rm s}\, k_{\rm fluct}^2$. If this term scales with t^2 the critical exponent of $k_{\rm fluct}$ must be *smaller* than 1. This implies that $\varrho_{\rm coh} \propto k_{\rm coh} = k_{\rm fluct}$ is asymptotically large compared to $\varrho_0 \sim |t|$. Therefore,

$$\varrho_{\rm s} = \varrho_0 + \varrho_{\rm coh} \sim \varrho_{\rm coh} \sim k_{\rm fluct} \;. \tag{39}$$

The last step follows from Eqs. (11) and (36). This leads to

$$\langle g_{\rm fluct}\rangle \propto \varrho_s\, k_{\rm fluct}^2 = (\varrho_0 + \varrho_{\rm coh})\, k_{\rm fluct}^2 \sim k_{\rm fluct}^3 \;. \tag{40}$$

The scaling assumption $\langle g_{\rm fluct}\rangle \sim \langle g_{\rm L}\rangle \sim t^2$ implies then

$$k_{\rm fluct} \sim |t|^{2/3} \;. \tag{41}$$

According to Eq. (39) the superfluid density has the same critical exponent, $\varrho_s \sim k_{\rm fluct} \sim |t|^{2/3}$.

To summarize: The assumption of the coherent comotion (leading to $\varrho_{\rm coh} \propto k_{\rm coh} = k_{\rm fluct}$) and the scaling of the fluctuation part ($\langle g_{\rm fluct}\rangle \propto \varrho_{\rm coh}\, k_{\rm fluct}^2 \propto k_{\rm fluct}^3$) with the Landau part ($\langle g_{\rm L}\rangle \sim t^2$) determines the critical exponent $\nu = 1/3$.

IV. CORRELATION LENGTH

A. Derivation

We derive the correlation length in our effective GL model. For this purpose we add a standard coupling term $-h\cdot\Psi$ to the integrand of the energy functional (23) where h is a fictitious external field. The variation of the energy functional with respect to the Ψ field leads to the field equation

$$\frac{a_0\,\lambda_{\rm c}^2}{2\pi f\,|t|^{1/3}}\,\Delta\Psi - 2\,\frac{\Delta c_P}{k_{\rm B}}\left(t\,\frac{\Psi}{f} + \frac{v\,|\Psi|^2\,\Psi}{f^2}\right) + h(\mathbf{r}) = 0 \;. \tag{42}$$

A small external field $h(\mathbf{r}) = \delta h_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$ will change the field from its equilibrium value $\langle\Psi\rangle$ to $\Psi = \langle\Psi\rangle + \delta\Psi_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$. We insert these expressions for $h(\mathbf{r})$ and $\Psi(\mathbf{r})$ into the field equation. In zeroth order in $\delta\Psi$ this yields the equilibrium value $\langle|\Psi|^2\rangle$. In first order in $\delta\Psi$ we obtain

$$\left[\frac{a_0\,\lambda_{\rm c}^2}{2\pi f\,|t|^{1/3}}\,k^2+2\,\frac{\Delta c_P}{k_{\rm B}}\left(\frac{t}{f}+3\,\frac{v\,\langle|\Psi|^2\rangle}{f^2}\right)\right]\delta\Psi_{\mathbf k}=\delta h_{\mathbf k}\ . \tag{43}$$

This susceptibility of the system with respect to an external field is given by

$$\chi(k)=\frac{\delta\Psi_{\mathbf k}}{\delta h_{\mathbf k}}=\frac{\chi(0)}{1+k^2\,\xi^2}\ . \tag{44}$$

The k dependence displayed in the last expression follows from Eq. (43). It implies

$$h(\mathbf r)=h_0\,\delta(\mathbf r)\quad\longrightarrow\quad\delta\Psi(\mathbf r)\propto\exp(-r/\xi)\ . \tag{45}$$

This means that the quantity ξ introduced in Eq. (44) defines the *correlation length*. Comparing Eqs. (43) and (44) yields

$$\xi^2=\frac{\frac{a_0\,\lambda_{\rm c}^2}{2\pi\,|t|^{1/3}}}{2\,\frac{\Delta c_P}{k_{\rm B}}\left(t+3\,\frac{v\,\langle|\Psi|^2\rangle}{f}\right)}\ . \tag{46}$$

Inserting the equilibrium values $\langle|\Psi|^2\rangle=0$ for $t<0$ and $\langle|\Psi|^2\rangle=f\,|t|/v$ for $t<0$ we obtain

$$\xi(t)=\begin{cases}\xi^+(t)=\xi_0\,|t|^{-2/3} & (t>0)\\ \xi^-(t)=\xi_0\,|t|^{-2/3}/\sqrt{2} & (t<0)\end{cases} \tag{47}$$

where

$$\xi_0=\frac{\lambda_{\rm c}}{\sqrt{4\pi}}\sqrt{\frac{a_0}{\Delta c_P/k_{\rm B}}}\approx 1.53\ \text{Å}\ . \tag{48}$$

For the numerical value we used Eqs. (24) and (25). The result is independent of the parameter f.

The Ψ theory by GS leads to Eqs. (47) and (48), too, but with an additional factor $[(3+M)/3]^{1/2}$ for ξ^+ and $[(3+M)/(3+3M)]^{1/2}$ for ξ^-; here M is an extra parameter. For $M=0$ the results by GS and the present model coincide. For the favored M value by GS ($M=0.6\pm0.3$ according to Eq. (20) in Ref. 4) the GS values differ from ours by about 10%.

The validity of the result Eq. (47) is restricted to the temperature range $|t|\lesssim 0.01$, Eq. (28). Outside this asymptotic region we expect the following behavior: For $|t|>0.2$ we find (Fig. 1) $\varrho_{\rm s}\approx\varrho_0$ corresponding to $R\approx1$ in Eq. (22). This implies that in the temperature range from $|t|=0.01$ to $|t|=0.2$ the correlation length will change its behavior from $\xi^-\sim|t|^{-2/3}$ to $\xi^-\sim|t|^{-1/2}$.

B. Critical fluctuations

A standard GL ansatz (in $d=3$ dimensions) breaks down for $|t|\to0$ because the ratio of the fluctuating to the equilibrium field diverges. In this section we will show that this problem does not exist in our effective GL model. This is due to the scaling invariance (Sec. III C) of our energy functional. The following discussion proceeds along the lines followed in a standard GL approach (see, for example, Ref. 12).

Equation (43) determines the deviation $\delta\Psi$ from the equilibrium due to an external field. Deviations from equilibrium occur also due to fluctuations. The fluctuations of the thermodynamic potential are of the size $\delta G=\mathcal O(k_{\rm B}T)$. This determines the size of the thermal fluctuations $\delta\Psi_{\rm therm}$ of the order parameter field.

We write $\Psi=\langle\Psi\rangle+\delta\Psi$ and expand $G_{\rm GL}=\langle G_{\rm GL}\rangle+\delta G_{\rm GL}$ into powers of $\delta\Psi$. The term linear in $\delta\Psi$ vanishes because we expand around the equilibrium value. Therefore, $\delta G_{\rm GL}$ starts with the quadratic term:

$$\frac{\delta G_{\rm GL}}{k_{\rm B}T_\lambda}=\int d^3r\,\frac12\,\frac{\partial^2 g}{\partial|\Psi|^2}\,|\delta\Psi|^2\ . \tag{49}$$

Here g denotes the integrand in Eq. (23). The derivative has to be taken at the equilibrium value:

$$\begin{aligned}\frac{\partial^2 g}{\partial|\Psi|^2}&=2\,\frac{\Delta c_P}{k_{\rm B}}\left(\frac{t}{f}+\frac{3\,v\,\langle|\Psi|^2\rangle}{f^2}\right)\\ &=\frac{2}{f}\,\frac{\Delta c_P}{k_{\rm B}}\,|t|\cdot\begin{cases}1 & (t>0)\\ 2 & (t<0)\end{cases}\ .\end{aligned} \tag{50}$$

The typical range of a thermal fluctuation $\delta\Psi_{\rm therm}$ is given by the correlation length $\xi(t)$. For the following estimate we may, therefore, replace the $\int d^3r$ in Eq. (49) by ξ^3. Using this approximation and $\delta G_{\rm GL,\,therm}=\mathcal O(k_{\rm B}T_\lambda)$, Eq. (49) becomes

$$\frac{\delta G_{\rm GL}}{k_{\rm B}T_\lambda}\approx\frac{\xi^3}{2}\,\frac{\partial^2 g}{\partial|\Psi|^2}\,|\delta\Psi_{\rm therm}|^2=\mathcal O(1)\ . \tag{51}$$

Inserting Eq. (50) and omitting factors of the order 1 we obtain

$$|\delta\Psi_{\rm therm}|^2=\frac{\mathcal O(1)}{|t|\,\xi(t)^3}\sim|t|\,. \tag{52}$$

For the last step we used Eq. (47), $\xi\sim|t|^{-2/3}$. The applicability of a GL approach is restricted to temperatures where the fluctuations are small (or at least not large) compared to the mean value of the field. The relevant ratio is

$$\frac{|\delta\Psi_{\rm therm}|^2}{\langle|\Psi|^2\rangle}\sim\begin{cases}\text{const.} & \text{present model}\\ |t|^{-1/2} & \text{GL}\end{cases}\ . \tag{53}$$

In a standard GL approach the ratio diverges for $|t|\to0$; this excludes the application of the model in the asymptotic region. In our effective GL model the ratio remains finite for $|t|\to0$. This is the same behavior as in a standard GL approach in $d=4$ dimensions.

V. TRANSITION TEMPERATURE IN HELIUM FILM

We consider a helium film on the y-z-plane extending from $x=0$ to $x=D$. We look for a nonzero solution

$\Psi(x)$ of the field equation that obeys the boundary conditions

$$\Psi(0) = \Psi(D) = 0\ . \tag{54}$$

This implies that a nonvanishing $\Psi(x)$ is inhomogeneous. Due to the kinetic energy term in Eq. (23) such a nonvanishing $\Psi(x)$ becomes possible only at a temperature $T_\lambda(D)$ below $T_\lambda = T_\lambda(\infty)$. We calculate this transition temperature $T_\lambda(D)$ as a function of the film thickness D.

In an experiment the plane $x = 0$ might be a solid wall, and the plane $x = D$ may define the free surface of the helium film. In this situation one has a solid layer (thickness d) of helium at the wall so that the first boundary condition actually reads $\Psi(d) = 0$. This point amounts to replacing D by $D - d$ and is ignored in the following. For a discussion of the appropriate boundary condition at the free surface we refer to GS[3].

We consider the field equation Eq. (42) for $h(\mathbf{r}) = 0$ and $t < 0$. We introduce the quantity ξ_0 of Eq. (48) and specialize to the one dimension of interest:

$$\xi_0^2\,|t|^{-4/3}\,\frac{d^2\Psi(x)}{dx^2} + \Psi\left(1 - \frac{v\,|\Psi|^2}{f\,|t|}\right) = 0\ . \tag{55}$$

For determining the *onset* of a nonvanishing Ψ, the Ψ^3 term may be neglected. The solution of the remaining differential equation is $\Psi = A\,\sin(x\,|t|^{2/3}/\xi_0 + \alpha)$ where $|A|^2 \ll f|t|/v$. The boundary condition $\Psi(0) = 0$ implies $\alpha = 0$, and $\Psi(D) = 0$ leads to $D|t|^{2/3}/\xi_0 = n\pi$ where n is a positive integer. We are looking for the highest temperature T or the smallest $|t|$ value ($t < 0$) for which such a solution exists. That means we have to take the $n = 1$ solution for which the boundary condition yields

$$\pi\xi_0\,|t|^{-2/3} = D\,. \tag{56}$$

This defines the maximum T value for a nontrivial solution. For a more elaborate discussion of the underlying mathematics we refer to Ref. 1. Using $t(D) = [T_\lambda(D) - T_\lambda(\infty)]/T_\lambda(\infty)$ we resolve Eq. (56) for $T_\lambda(D)$:

$$\begin{aligned} T_\lambda(D) &= T_\lambda(\infty)\left[1 - \left(\frac{\pi\,\xi_0}{D}\right)^{3/2}\right] \\ &\approx T_\lambda(\infty)\left[1 - \left(\frac{4.81\,\text{Å}}{D}\right)^{3/2}\right]\ . \end{aligned} \tag{57}$$

Here $T_\lambda(\infty)$ denotes the transition temperature of the bulk system. The numerical value follows from Eq. (48).

As discussed by GS, the experimental and theoretical situation in a helium film is more delicate. According to GS[4] there is a caloric transition, and at a slightly lower temperature another transition to superfluidity. The condition for the second transition is somewhat different from that one discussed in this section. As shown by Kosterlitz and Thouless[13], superfluidity sets in when the superfluid density (averaged over the film thickness) exceeds a certain minimum value. Evaluating this condition in our model leads to an expression of the form (57), too, but with a slightly different length parameter (instead of $\pi\,\xi_0$).

For the comparison with the experimental findings we refer again to GS[4]. One finds indeed transition temperature shifts that are proportional to $D^{-3/2}$, and length scales similar to the calculated one.

VI. DENSITY PROFILES AT A BOUNDARY

A. Introduction

We consider a helium film with a thickness D much larger than the correlation length ξ. At the wall the superfluid density will raise from zero to its bulk value $\varrho_{\rm s}(\text{bulk})$ (for simplicity we ignore again the solid layer at the wall). Similarly, the superfluid density will fall off again towards the free surface.

The following discussion is restricted to the density profile at one boundary. This means that we determine how the density profile

$$f(x) = \frac{\varrho_{\rm s}(x)}{\varrho_{\rm s}(\text{bulk})} \tag{58}$$

raises from zero at $x = 0$ to its bulk value. This corresponds to the boundary conditions

$$f(0) = 0 \quad \text{and} \quad f(\infty) = 1\ . \tag{59}$$

Third sound measurements determine the value $\overline{\varrho_{\rm s}}$ of the superfluid density averaged over the film thickness D. By $\overline{\varrho_{\rm s}} = \varrho_{\rm s}(\text{bulk})(D - 2\,\xi_{\rm heal})/D$ one defines the *healing length* $\xi_{\rm heal}(t)$ (for a more comprehensive discussion see § 41 in Ref. 7). For a given profile $f(x)$ this healing length may be calculated by

$$\xi_{\rm heal} = \lim_{a\to\infty}\left(a - \int_0^a dx\, f(x)\right)\ . \tag{60}$$

We approach the question of the density profile in two different ways:

1. In the considered asymptotic regime ($|t| \lesssim 0.01$) the density $\varrho_{\rm coh}$ is the predominant contribution to the superfluid density. The particles forming $\varrho_{\rm coh}$ have a distribution of momenta up to about $k_{\rm coh}$. This leads to a healing length $\xi_{\rm heal,1} \sim 1/k_{\rm coh} \sim |t|^{-2/3}$.

2. We solve the field equation (55) with the boundary conditions Eq. (59). This yields the profile of condensate density ϱ_0, Eq. (30), and to a healing length $\xi_{\rm heal,2} \sim \xi_0\,|t|^{-2/3}$.

The composition (5) of the superfluid density leads thus to the prediction that the boundary profiles of the densities $\varrho_{\rm s}$ and ϱ_0 are different.

B. Profile of the superfluid density

For $|t| \ll 1$ the superfluid density may be written as

$$\varrho_{\rm s} \approx \varrho_{\rm coh} \propto \int_0^{k_{\rm coh}} dk\, k^2 \,\langle n_k\rangle\, |\varphi_{\mathbf k}(\mathbf r)|^2 \ . \qquad (61)$$

The $\varphi_{\mathbf k}(\mathbf r)$ are the real single particle functions of low-lying noncondensed states. For $|t| \ll 1$ the occupation numbers (6) are $\langle n_k\rangle \approx 1/(\kappa^2+\tau^2)$. Because of $\tau^2 \ll \kappa_{\rm coh}^2$ we may write $\kappa^2/(\kappa^2+\tau^2) \approx 1$. At the boundary plane $x=0$ all single particle functions must vanish, i. e. $|\varphi_{\mathbf k}(\mathbf r)|^2 \propto \sin^2(k_x x)$. Since $k^2 = k_x^2 + k_y^2 + k_z^2$ is limited by $k_{\rm coh}^2$, we assume that k_x^2 is limited by $k_{\rm coh}^2/3$. Putting these (partly approximate) steps together yields

$$\varrho_{\rm s}(x) \propto \int_0^{k_{\rm coh}/\sqrt{3}} dk_x\, \sin^2(k_x x)$$
$$= \frac{k_{\rm koh}}{2\sqrt{3}} - \frac{\sin\big(2k_{\rm coh}x/\sqrt{3}\big)}{4x} \ . \qquad (62)$$

Including all constants, the first term on the r. h. s. equals the bulk value; it is obtained as the limit of Eq. (62) for large x. Therefore, we may read off the density profile:

$$\frac{\varrho_{\rm s}(x)}{\varrho_{\rm s}({\rm bulk})} = 1 - \frac{\sin\big(2k_{\rm coh}x/\sqrt{3}\big)}{2k_{\rm coh}x/\sqrt{3}} = f_1(x) \ . \qquad (63)$$

We evaluate the healing length (60) for this profile,

$$\xi_{\rm heal,\,1} = \frac{\pi\sqrt{3}}{4k_{\rm coh}} = \frac{\sqrt{3}}{2\,\zeta(3/2)}\,\frac{\lambda_{\rm c}}{a_0}\,|t|^{-2/3} = 0.82\,\text{Å}\,|t|^{-2/3} \ . \qquad (64)$$

We used Eqs. (7), (13) and (14) for the second step, and Eqs. (24) and (25) for the numerical value.

C. Profile of the condensate density

Using the dimensionless quantities $\psi = \Psi(x)/\Psi({\rm bulk})$ with $\Psi({\rm bulk}) = \sqrt{f|t|/v}$ and $y = (x/\xi_0)\,|t|^{2/3}$, the field equation (55) reads

$$\psi''(y) + \psi(y)\left[1-\psi(y)^2\right] = 0 \ . \qquad (65)$$

The boundary conditions Eq. (59) become $\psi(0)=0$ and $\psi(\infty)=1$. The solution of Eq. (65) obeying these boundary conditions is $\psi(y) = \tanh(y/\sqrt{2})$. Returning to the original variables we obtain the profile of the condensate density (30),

$$\frac{\varrho_0(x)}{\varrho_0({\rm bulk})} = \tanh^2\left(\frac{x}{\sqrt{2}\,\xi_0\,|t|^{-2/3}}\right) = f_2(x) \ . \qquad (66)$$

Inserting this profile into Eq. (60) yields the healing length

$$\xi_{\rm heal,\,2} = \sqrt{2}\,\xi_0\,|t|^{-2/3} = 2.16\,\text{Å}\,|t|^{-2/3} \ . \qquad (67)$$

This differs by a factor of 2 from the correlation length $\xi^-(t)$.

D. Discussion

The calculated density profiles of the superfluid and condensate density, Eqs. (59) and (66), respectively, are depicted in Fig. 2. These results refer —as most results in this paper— to the asymptotic region $|t| \lesssim 0.01$, Eq. (28).

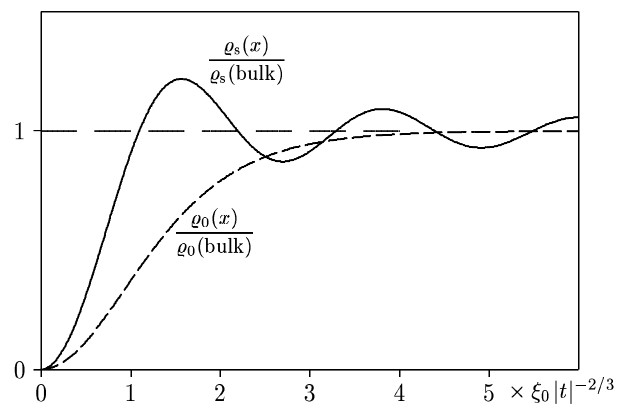

FIG. 2. Superfluid density $\varrho_{\rm s}(x)$ and the condensate density $\varrho_0(x)$ as functions of the distance x from a boundary. The distance x is measured in units of $\xi_0\,|t|^{-2/3}$. For large x the densities approach their bulk values.

The occurrence of different profiles does not alter the calculation of Sec. V: For a superfluid flow we must have a nonvanishing order parameter field. The condition obtained in Sec. V is sufficient to accommodate both, ϱ_0 and $\varrho_{\rm coh}$, within the film.

The experimental result for the healing length cited in Eq. (41-47) of Ref. 7 is $\xi_{\rm heal}^{\rm exp} \approx 1.55\,\text{Å}\,|t|^{-2/3}$. The question is which one of the calculated results, (64) or (67), should be compared with the experimental one. Both results scale correctly. Relative to the experimental prefactor, one theoretical prefactor is about 50% to large, in the other case about 50% too small. Because of $\varrho_{\rm coh}({\rm bulk}) \gg \varrho_0({\rm bulk})$ (for $|t| \lesssim 0.01$) one might be inclined to consider the correlation length (64) as the relevant quantity. However, (i) the macroscopic Ψ field defines the macroscopic phase field necessary for superfluidity, and (ii) the simple averaging procedure (60) might not be appropriate for a profile like $f_1(x)$. In view of this more involved situation, we restrict ourselves to the statement that the theoretical results are in a reasonable vicinity of the experimental one.

A direct experimental verification of the predicted profiles shown in Fig. 2 appears hardly feasible. The $\varrho_{\rm s}$ profile may have some influence on third sound calculations the results of which might then be tested. The condensate density is already in the bulk system[10] a hard to access quantity.

The different profiles calculated here are obviously in variance with the GS approach. It is a future task to find differing predictions of our model that might be readily subject to an experimental test.

VII. CONCLUDING REMARKS

We have proposed an effective Ginzburg-Landau model that is compatible with the bulk properties of liquid helium. We have investigated the basic properties of this model and its most simple applications for finite geometries.

Our model is based on the IBG and the assumption of a coherent comotion of condensed and noncondensed particles. The occurrence of fractional powers in the energy functional can be made plausible on this basis. The foundation of our model is thus different from the Ψ theory by GS[2–4].

The results obtained so far are mostly similar to that of the Ψ theory by GS[2–4]. This is to be expected for other applications, too. The results for the density profiles at a boundary are, however, novel and in variance with the Ψ theory.

There is a variety of further problems that might be treated in our approach (and that have already been treated in the Ψ theory), like the influence of external fields or of impurities (ions) and the description of superfluid flow in various geometries.

The Ψ theory has been generalized[4] in order to incorporate (i) time-dependent problems in the field equation, (ii) dissipation effects and (iii) problems with a nonvanishing normal velocity. These generalizations correspond to additional terms in the functional of the thermodynamic potential. All these generalization can be introduced in our model in a straight-forward way, too.

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