

A Model for the λ -Transition of Helium[†]

Torsten Fließbach*

Fachbereich Physik, University of Siegen, 57068 Siegen, Germany

Guided by the analogy to the Bose-Einstein condensation of the ideal Bose gas (IBG) we propose a new model for the λ -transition of liquid helium. Deviating from the IBG our model uses phase ordered and localized single-particle functions. This means that finite groups of particles are assumed to be phase-locked. These phase correlations can be related to the singularity at the transition point and to the occurrence of the superfluid density.

The model leads to the following results:

1. A possible explanation of the logarithmic singularity of the specific heat.
2. A characteristic functional form for the superfluid density which yields excellent fits to the experimental data.
3. A quantitative prediction of a small but nonzero entropy content of the superfluid component.

I. INTRODUCTION

In 1938 London¹ presented his famous suggestion about the connection between the λ -transition in ⁴He and the Bose-Einstein condensation in the ideal Bose gas² (IBG). About ten years later this point of view was confirmed experimentally by showing the absence of such a transition in ³He. Theoretically it was supported by Feynman³ who argued that London's view was essentially correct. Furthermore, the microscopic IBG provides a means of understanding⁴ some properties of He II. The actual properties of helium are, however, in many respects quite different from the IBG. In view of this we propose a model which is a modification of the IBG invented such that relevant equilibrium properties of the real system are reproduced.

The IBG can be defined by the microscopic states

$$\Psi_{\text{IBG}}(\mathbf{r}_j, n_k) = \mathcal{S} \prod_{\mathbf{k}} [\varphi_{\mathbf{k}}]^{n_k}, \quad (1)$$

which are the symmetrized (\mathcal{S}) product states of single-particle functions (s.p.f.) $\varphi_{\mathbf{k}}$ of momentum $\hbar\mathbf{k}$. The states (1) depend on the coordinates \mathbf{r}_j (where $j = 1, \dots, N$) of the atoms and on the parameters $n_k = n_{\mathbf{k}}$. The temperature dependent expectation values $\langle n_k \rangle$ of the occupation numbers display the phase transition of the IBG.

Our model is based on two major assumptions:

1. It starts with an IBG-related *guess* for the form of the many-body states. As in Eq. (1), these states depend on occupation numbers n_k , but beyond Eq. (1) they contain correlations due to the use of localized and phase ordered single-particle functions.

2. The phase transition is introduced phenomenologically by using the IBG values for $\langle n_k \rangle$. Due to this feature our model is closely related to the IBG and will therefore be called *almost ideal Bose gas model*, or AIBG.

The basic idea of the AIBG for introducing the relevant correlations is the following: Normally exponential functions $\varphi_{\mathbf{k}} \propto \exp(i\mathbf{k} \cdot \mathbf{r}_j)$ are used for the s.p.f. in Eq. (1). Alternatively one may consider sinus functions which depend on the component $x_j = \hat{\mathbf{x}} \cdot \mathbf{r}_j$ of \mathbf{r}_j as follows:

$$\varphi_{\mathbf{k}} \propto \sin(qx_j + \phi_j). \quad (2)$$

Here $\hat{\mathbf{x}}$ is a unit vector in an arbitrary direction, and $q = \hat{\mathbf{x}} \cdot \mathbf{k}$. For these s.p.f. we introduce the concept of *phase ordering* (p.o.):

$$\phi_j = \phi_0 \quad (\text{phase ordering}). \quad (3)$$

Phase ordered (p.o.) s.p.f. with the same q (but in general different \mathbf{k}) are correlated because their squares $|\varphi_{\mathbf{k}}(\mathbf{r}_j)|^2$ exhibit common extrema. Thus p.o. leads to an extra spatial correlation between all atoms with the same q . Such correlations may be important if many atoms share the same momentum. Therefore p.o. is a potentially decisive correlation near the transition point in an IBG-like model.

A p.o. may be introduced by physical (or Dirichlet) boundary conditions at the walls of the macroscopic volume V . Then the resulting effects are surface effects and vanish like $V^{-1/3}$ for $V \rightarrow \infty$. In contrast to this, we obtain *finite* effects by assuming s.p.f. $\varphi_{\mathbf{k}}$ (l.p.o.) which are not only phase ordered (p.o.) but also *localized* (l.). For this purpose the volume V is thought to be divided into V/V_0 finite boxes of size V_0 . The $\varphi_{\mathbf{k}}$ (l.p.o.) are then the s.p.f. which are confined to one of these boxes and subject to Dirichlet boundary conditions. For these s.p.f. we construct a state of the form (1):

$$\Psi_{\text{M}}(\mathbf{r}_j, n_k, N_0) = \mathcal{S} \prod_{\text{vol}} \prod_{\mathbf{k}} [\varphi_{\mathbf{k}}(\text{l.p.o.})]^{n_k}. \quad (4)$$

The product runs over all momenta and all finite volumes. The states Ψ_{M} depend on the additional parameter N_0 which denotes the number of atoms in V_0 . For $N_0 \rightarrow \infty$ these states reduce to IBG states. As a surface effect of the finite boxes the l.p.o. introduces correlations of the order $N_0^{-1/3}$.

As it stands, the states Ψ_{M} cannot be used together with a realistic Hamiltonian H . Therefore, we multiply them by a suitable Jastrow factor,

$$\Psi(\mathbf{r}_j, n_k, N_0) = F\Psi_M, \quad F = \prod_{i<j} f(r_{ij}). \quad (5)$$

We use Jastrow functions $f(|\mathbf{r}_i - \mathbf{r}_j|)$ which have been determined⁵ by minimizing the energy $E_0 = \langle F|H|F \rangle$.

We have now sketched the basic idea of the AIBG, namely the modification of the IBG due to a phase ordering of the s.p.f. The resulting model is rather close to the IBG; in spite of some formal similarity of the underlying states it is not related to the quasi-particle model (note that $\sum n_k = N$ for Eq. (5)). In Sec. II the AIBG will be defined in detail. This includes the evaluation of the energy

$$E(n_k) = \langle F\Psi_M|H|F\Psi_M \rangle \quad (6)$$

Evaluated with $\langle n_k \rangle_{\text{IBG}}$ this energy yields a logarithmic singularity (Sec. III).

Since our approach is based on the IBG (above point 2), the critical exponent β of the model condensate density has the value $1/2$. This differs from the critical exponent $\nu \simeq 1/3$ of superfluid density ρ_s . Therefore, ρ_s cannot be identified with the square of the condensate wave function. In the AIBG we argue that due to p.o. also noncondensed particles contribute to ρ_s ; in any IBG-like model ($\beta = 1/2$) such a contribution will be required in order to account for the experimental value $\nu \simeq 1/3$. This contribution leads to the peculiar consequence that the superfluid component has a nonvanishing entropy $S_s \neq 0$. A prediction for $S_s \neq 0$ will either falsify the model right away, or provide a crucial test of the model. Therefore we extend the model in Sec. V to VI such that ρ_s and S_s can be calculated.

Section IV shows how the localization of the s.p.f. can be reconciled with a macroscopic p.o. and introduces generalized s.p.f. which are able to describe the coherent motion of noncondensed particles with the condensate. Section V presents the model expression for ρ_s and shows that it yields excellent fits to the experimental data. With the model parameters fixed by this fit we obtain in Sec. VI the crucial model prediction for $S_s \neq 0$. This prediction is compared to the available experimental data; it is found that the prediction is at the border of present-day experimental detectability.

II. THE ALMOST IDEAL BOSE GAS MODEL (AIBG)

For a complete definition of our model we define the l.p.o. s.p.f. (Sec. II A), calculate the energy E (Sec. II B) and specify the assumptions about the expectation values of the parameters (Sec. II C).

A. Localized, phase ordered single-particle functions

The l.p.o. s.p.f. in Eq. (4) are formally defined as follows: The macroscopic volume V is divided into V/V_0

cubic boxes of size V_0 . Within each box we define orthonormalized s.p.f. which obey Dirichlet boundary conditions at the walls:

$$\varphi_{\mathbf{k}}(\text{l.p.o.}) = \begin{cases} \sqrt{\frac{8}{V_0}} \prod_{i=1}^3 \sin(k_i x_i) & \text{for } \mathbf{r} \in V_0, \\ 0 & \text{elsewhere.} \end{cases} \quad (7)$$

The Cartesian components k_i of \mathbf{k} are restricted to the discrete values q_n ,

$$q_n = n \cdot \Delta k, \quad n = 1, 2, \dots, \quad \Delta k = \pi/V_0^{1/3}. \quad (8)$$

The many-body states of the AIBG are now defined by Eq. (5) with Eqs. (4) and (7).

This construction is chosen for simplicity. It contains artificial aspects like the cubic shape and the identical size of the boxes. The underlying physical picture is that there are finite regions (of various shapes and sizes) in which the atoms lower their free energy (see Sec. IV A) by assuming p.o. in just *one* direction. For macroscopic physical quantities we have to average (explicitly or implicitly) over such regions, or over the boxes used for Eq. (7). Formally we perform this averaging by replacing the discrete momentum sums by integrals:

$$\sum_{q_n} \dots \implies \int_q \dots = \frac{1}{\Delta k} \int_0^\infty dq \dots \quad (9)$$

The justification of the lower integral bound will be discussed in Sec. IV B. The averaging over boxes of finite size and shape restores also the translational and rotational invariance of the system as a whole. The occupation number for $\varphi_{\mathbf{k}}$ will therefore depend only on k but not on the direction of \mathbf{k} .

B. Energy

We evaluate the energy Eq. (6) for the Hamiltonian

$$H = - \sum_i \frac{\hbar^2}{2m} \Delta_i + \sum_{i<j} u(r_{ij}) \quad (10)$$

with a realistic atom-atom potential $u(r)$, for example of Lennard-Jones type.

The ground state (g.s.) may be approximated⁵ by $\Psi_0 = F$. The corresponding g.s. energy E_0 can be written as

$$E_0(V, N) = \langle F|H|F \rangle = \frac{N^2}{2V} \int d^3r \tilde{u}(r) g_0(r), \quad (11)$$

where $g_0(r)$ is the pair correlation function for Ψ_0 and $\tilde{u}(r)$ is given by

$$\tilde{u}(r) = u(r) - \frac{\hbar^2}{2m} \ln f(r). \quad (12)$$

Sensible results for E_0 and $g_0(r)$ are obtained⁵ by a simple ansatz for $f(r)$.

The following evaluation of the energy E for $\Psi = F\Psi_M$ is based on the fact that the main correlations are determined by F . The additional correlations due to $\Psi_M \neq 1$ are small, they are of the order $N_0^{-1/3} \ll 1$ for the relevant p.o. effects. The pair correlation function $g(r)$ for $\Psi = F\Psi_M$ can therefore be written as

$$g(r, n_k) = g_0(r) + \Delta g(r, n_k), \quad \Delta g \ll 1. \quad (13)$$

We relate Δg to the pair correlation function g_M of Ψ_M . (An explicit expression for g_M and details of the following derivation are given in Ref. 6). Writing $g_M = 1 + \Delta g_M$ it follows that $\Delta g \rightarrow 0$ for $\Delta g_M \rightarrow 0$, or $\Delta g \propto \Delta g_M$. To lowest order in $f(r)$ Eq. (5) yields

$$\Delta g(r, n_k) \simeq f(r)^2 \Delta g_M(r, n_k). \quad (14)$$

An alternative approximation is $\Delta g \simeq g_0 \Delta g_M$; the difference to Eq. (14) marks the uncertainty in this approximation. For our purpose both approximations will turn out to be nearly equivalent. The relation between Δg and Δg_M cannot be evaluated exactly; for a discussion of this problem see Ref. 7.

The potential energy for $F\Psi_M$ is $E_{\text{pot}} = (N^2/V) \cdot \int d^3r g u$. The kinetic energy can be divided into two parts: The Jastrow factors are taken into account by replacing u in E_{pot} by \tilde{u} , as in Eq. (11). The kinetic contributions due to $\Psi_M \neq 1$ lead to $\sum \varepsilon_k n_k$ where $\varepsilon_k = \hbar^2 k^2 / 2m$. Here m is an effective mass (Ref. 6) but for simplicity we do not introduce a new symbol. We emphasize that $m \neq 0$ for $k \rightarrow 0$. This can be related to the fact that the static structure function $S(k, T)$ is finite for $k \rightarrow 0$ and $T \simeq T_\lambda$. The complete result reads

$$\begin{aligned} E(V, N, n_k) &= \langle F\Psi_M | H | F\Psi_M \rangle \\ &= \frac{N^2}{2V} \int d^3r g(r, n_k) \tilde{u}(r) + \sum_{\mathbf{k}} \varepsilon_k n_k \\ &= E_0(V, N) + E_M(V, N, n_k). \end{aligned} \quad (15)$$

The energy $E_M = E - E_0$ is obtained by inserting Eqs. (13), (14) and (11):

$$\begin{aligned} E_M(V, N, n_k) &\simeq \sum_{\mathbf{k}} \varepsilon_k n_k \\ &+ \frac{N^2}{2V} \int d^3r f(r)^2 \Delta g_M(r, n_k) \tilde{u}(r). \end{aligned} \quad (16)$$

The decisive contributions in Δg_M are long-ranged because the low momenta dominate near the transition. For the intended evaluation of the critical behaviour we may therefore use the approximation

$$\Delta g_M(r, n_k) \simeq \Delta g_M(0, n_k). \quad (17)$$

This yields

$$E_M(V, N, n_k) = \sum_{\mathbf{k}} \varepsilon_k n_k + \frac{N}{2} \frac{w_0}{v} \Delta g_M(0, n_k), \quad (18)$$

where $v = V/N$ and

$$\begin{aligned} \frac{w_0}{v} &\simeq \frac{1}{v} \int d^3r f(r)^2 \tilde{u}(r) \simeq \frac{1}{v} \int d^3r g_0(r) \tilde{u}(r) \\ &\simeq \frac{2E_{\text{g.s.}}}{N} = -14.3 \text{ k}_B\text{K}. \end{aligned} \quad (19)$$

Using McMillan's parameters⁵, the first integral yields $-14.8 \text{ k}_B\text{K}$. Within the uncertainty of Eq. (14) it may be replaced by the second integral. This integral is the theoretical g.s. energy (11). By equating this energy to the experimental ground state energy $E_{\text{g.s.}}$ we fix the value of w_0/v .

Up to a constant, the energy E_M (18) can be expressed by

$$\begin{aligned} E_M &= \langle \Psi_M | H | \Psi_M \rangle \quad \text{with} \\ H_M &= - \sum_i \frac{\hbar^2}{2m} \Delta_i + w_0 \sum_{i < j} \delta(\mathbf{r}_i - \mathbf{r}_j). \end{aligned} \quad (20)$$

This means that within sensible approximations the energy E_M in $E = E_0 + E_M$ can be calculated as the expectation value of a simple product state Ψ_M with a simple Hamiltonian H_M . It should be noted that H_M is solely introduced for evaluating E_M , it cannot serve as a model Hamiltonian in other respects. In particular the stability of the system is guaranteed by the part E_0 of the total energy $E = E_0 + E_M$; the corresponding part of the Hamiltonian is not contained in H_M .

The attractive zero-range force in H_M has the following meaning: The l.p.o. implies an enhanced probability of finding two atoms near together. Due to the Jastrow factors in Eq. (5) this enhancement is effective only in the range where the realistic interaction is attractive; the strength of this attraction is measured by w_0/v . A zero-range force can be used because for g_M the enhancement is approximately r -independent up to a few Å.

The evaluation of Eq. (20) is straightforward. The two-body matrix elements $w_{\mathbf{k}\mathbf{q}}$ of the δ -force with the s.p.f. (7) are given by

$$\begin{aligned} w_{\mathbf{k}\mathbf{q}} &= \langle \varphi_{\mathbf{k}} \varphi_{\mathbf{q}} | w_0 \delta(\mathbf{r}_1 - \mathbf{r}_2) | \varphi_{\mathbf{k}} \varphi_{\mathbf{q}} \rangle \\ &= \frac{w_0}{V_0} \prod_{i=1}^3 \left[1 + \frac{1}{2} \delta_{k_i q_i} \right] \\ &= \frac{w_0}{V_0} \left[1 + \frac{1}{2} \sum_{i=1}^3 \delta_{k_i q_i} + \dots \right]. \end{aligned} \quad (21)$$

The somewhat artificial construction Eq. (7) implies p.o. in 3 orthogonal directions. In contrast to this we assume that the p.o. is actually realized in just one direction. Therefore we omit products of δ -functions in the

last expression of Eq. (21) which originate from simultaneous p.o. in 2 or 3 directions. The remaining δ -terms describe the mutual correlation of all particle with the same momentum q in one direction. The number ν_q of these particles is

$$\nu_q = \sum_{k_2, k_3} n_k \quad \text{where} \quad k = |\mathbf{k}| = \sqrt{q^2 + k_2^2 + k_3^2}. \quad (22)$$

The obvious form of the correlation energy is $E_{\text{corr}} \propto \sum \nu_q^2$. The actual result follows from Eq. (20) with Eqs. (21) and (22):

$$E_{\text{corr}}(V, N, n_k) = \frac{3w_0}{2v} \frac{1}{N_0^{1/3}} \left[\frac{1}{N^{2/3}} \left(\sum_q \nu_q^2 + 2n_0\nu_0 \right) \right]. \quad (23)$$

For the replacement (9) of the sum by an integral we took into account a possible finite condensate fraction n_0/N . The index zero of n_0 and ν_0 stands for the lowest possible momentum value.

The complete expression for the energy E of Eq. (15) becomes

$$E(V, N, n_k) = E_0 + E_M = E_0 + E_{\text{IBG}} + E_{\text{corr}}. \quad (24)$$

Here $E_0 = E_0(V, N)$ is given by Eq. (11) and E_{IBG} by

$$E_{\text{IBG}}(V, N, n_k) = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}}. \quad (25)$$

The critical contribution in Eq. (24) is the correlation energy E_{corr} (23). In the evaluation of Eq. (20) leading to Eq. (24) we dismissed a constant term stemming from the constant in Eq. (21), the higher order terms ($\propto N_0^{-2/3}$ or N_0^{-1}) because they come from p.o. in more than one direction, and a term $\propto n_0^2$ because it is noncritical (that means eventually negligible near the transition point).

For the finite boxes the momentum sums have to be evaluated with $\Delta k = \pi/V_0^{1/3}$; the subsequent summation over all boxes yields, however, a factor V/V_0 . For plain momentum sums (without $\delta_{k_i q_i}$ -terms) this procedure can be abbreviated by performing the common summation with $\Delta k = \pi/V^{1/3}$. The final momentum sums in Eqs. (23) and (25) are performed in this common way. This implies that the occupation numbers n_k do no longer refer to a finite box (as in Eq. (4)) but to the macroscopic system (with V and N). In Eq. (23) this implies $\sum \nu_q^2 \propto N^{5/3}$ and an N -independent E_{corr}/N .

C. Statistical assumptions

In order to calculate the thermodynamic energy $E(T, V, N)$ from Eq. (24) we need the temperature dependent expectation values of the parameters of Ψ . Our model uses the IBG expression for $\langle n_k \rangle$ and treats N_0

as an adjustable constant. We discuss and specify these assumptions.

Any model of the λ -transition except the IBG introduces the phase transition phenomenologically. In the AIBG this phenomenological assumption is the use of the expectation values $\langle n_k \rangle_{\text{IBG}}$. This assumption can be made plausible (to some extent) by observing that the additional contribution E_{corr} in $E_M = E_{\text{IBG}} + E_{\text{corr}}$ is small in the sense

$$\frac{E_{\text{corr}}}{E_{\text{IBG}}} = \mathcal{O}(y) \quad \text{where} \quad y = \frac{|w_0/v| N_0^{-1/3}}{k_B T_\lambda} \ll 1. \quad (26)$$

The actual parameter values will yield $y = 0.13$, Eq. (42). For $N_0 \rightarrow \infty$ or $y \rightarrow 0$ one obtains $E_M \rightarrow E_{\text{IBG}}$ and thus $\langle n_k \rangle_{\text{IBG}}$. (The n_k -independent contribution $E_0(V, N)$ in E is without influence on the expectation values). Therefore, we expect $\langle n_k \rangle = \langle n_k \rangle_{\text{IBG}} [1 + \mathcal{O}(y)]$. The evaluation of E_{corr} with $\langle n_k \rangle_{\text{IBG}}$ is then valid up to first order in y ; it is this contribution which yields the logarithmic singularity. Another argument for using the $\langle n_k \rangle_{\text{IBG}}$ is that the additional term $\sum \nu_q^2$ correlates many s.p. states with many others, and that it will be without much influence on the occupation of a particular s.p. state.

The AIBG assumes expectation values $\langle n_k \rangle$ of the IBG-form,

$$\begin{aligned} \langle n_k \rangle &= \frac{1}{\exp[(\varepsilon_k - \mu)/k_B T] - 1} \\ &= \frac{1}{\exp(x^2 + \tau^2) - 1}. \end{aligned} \quad (27)$$

Here μ is the chemical potential and k_B is Boltzmann's constant. We have introduced the dimensionless quantities $\tau^2 = -\mu/k_B T$ and

$$x = \frac{\lambda |\mathbf{k}|}{\sqrt{4\pi}} \quad \text{with} \quad \lambda = \frac{2\pi\hbar}{\sqrt{2\pi m k_B T}}. \quad (28)$$

The transition temperature of the IBG is given by the following condition for the thermal wave length $\lambda = \lambda(T)$:

$$\lambda(T_\lambda) = [v \zeta(3/2)]^{1/3}, \quad (29)$$

where $\zeta(3/2) = 2.6124$ denotes Riemann's zeta function. In applying the AIBG to the real system we will identify T_λ with the actual transition temperature; formally this can be achieved by inserting a suitable effective mass $m = m(v)$ in λ of Eq. (28), Ref. 3. In the following we use the relative temperature

$$t = \frac{T - T_\lambda}{T_\lambda}. \quad (30)$$

For $t \geq 0$ the chemical potential is determined by the particle number condition $\sum n_k = N$. For $t \rightarrow 0$ this condition yields $\mu - \varepsilon_0 \rightarrow 0$ and thus $n_0 \rightarrow \infty$. For simplicity we rename $\mu - \varepsilon_0$ by μ ; it is this new μ which vanishes at $t = 0$ also for $\varepsilon_0 \neq 0$.

We are interested in the critical properties. Therefore we expand μ or, equivalently, τ for $|t| \ll 1$:

$$\tau(t) = \sqrt{\frac{-\mu}{k_B T}} = \begin{cases} at + bt^2 + \dots & (t > 0), \\ a'|t| + b't^2 + \dots & (t < 0). \end{cases} \quad (31)$$

This expansion is directly related to the model condensate fraction

$$\begin{aligned} \frac{\langle n_0 \rangle}{N} &= 1 - \frac{1}{N} \sum_{\mathbf{k}} \langle n_{\mathbf{k}} \rangle \\ &= \begin{cases} 0 & (t > 0), \\ f|t| + gt^2 + \dots & (t < 0). \end{cases} \end{aligned} \quad (32)$$

The coefficients f, g, \dots are determined by a', b', \dots ; in particular $f = 3/2 + 2\sqrt{\pi} a' / \zeta(3/2)$. The IBG yields finite values for a, b, \dots , and $a' = b' = \dots = 0$.

Deviating from the IBG, the AIBG admits a coefficient $a' \neq 0$ in Eq. (31) or $f > 3/2$ in Eq. (32). This deviation is introduced phenomenologically for the following reason: For $t > 0$ the evaluation of $\sum \nu_q^2$ with $\langle n_{\mathbf{k}} \rangle_{\text{IBG}}$ yields straightforwardly a logarithmic singularity. This singularity can be directly continued to $t < 0$ by admitting $a' \neq 0$; the singularity itself does not depend on the value for a' . The form of the expansion (31) is to some extent plausible because of its simplicity and symmetry. Physically $a' \neq 0$ means an energy gap of $a'^2 t^2 k_B T$ for the condensed s.p. state for $t < 0$.

For evaluating the thermodynamic energy we need also the expectation values of products of $n_{\mathbf{k}}$'s. (We do not need, however, the full statistical information contained in the density matrix). For such products the IBG yields

$$\langle n_{\mathbf{k}} n_{\mathbf{q}} \rangle = \langle n_{\mathbf{k}} \rangle \langle n_{\mathbf{q}} \rangle \quad (\mathbf{k} \neq \mathbf{q}). \quad (33)$$

Compared to the IBG, the AIBG states depend on one additional parameter, namely N_0 . In principle, this parameter (as well as the occupation numbers) should be determined from the condition of minimal free energy F yielding a temperature dependent expectation value $\langle N_0 \rangle$. Instead of this, we treat N_0 as an adjustable constant. Simplified estimates (Sec. IV A and Ref. 6) yield a sensible value for N_0 at $t = 0$.

III. LOGARITHMIC SINGULARITY

We evaluate the thermodynamic energy $E(T, V, N) = \langle E(V, N, n_{\mathbf{k}}) \rangle$. According to Eq. (24) it is of the form

$$E(T, V, N) = E_0(V, N) + E_{\text{IBG}}(T, V, N) + E_{\text{corr}}(T, V, N). \quad (34)$$

The term $E_0(V, N)$ is essential for a realistic compressibility but it does not contribute to the specific heat. The energy $E_{\text{IBG}} = \sum \varepsilon_{\mathbf{k}} \langle n_{\mathbf{k}} \rangle$ is calculated as in the IBG. Equations (23) and (33) yield for the decisive contribution E_{corr} ,

$$\frac{E_{\text{corr}}(T, V, N)}{N} = \frac{3w_0}{2v} \frac{1}{N_0^{1/3}} \frac{1}{N^{5/3}} \left[\sum_{\mathbf{q}} \langle \nu_{\mathbf{q}} \rangle^2 + 2 \langle n_0 \rangle \langle \nu_0 \rangle \right]. \quad (35)$$

Leaving away unessential constants we sketch how E_{corr} yields a logarithmic singularity: For $|t| \rightarrow 0$ and $x \rightarrow 0$ the occupation numbers (27) behave like $\langle n_x \rangle \sim 1/[x^2 + t^2]$ and $\langle \nu_q \rangle \sim \int dk k (k^2 + q^2 + t^2)^{-1} \sim \ln(q^2 + t^2)$. It follows that $\langle n_0 \rangle \langle \nu_0 \rangle \sim |t| \ln |t|$ and $\sum_{\mathbf{q}} \langle \nu_{\mathbf{q}} \rangle^2 \sim \int dq [\ln(q^2 + t^2)]^2 \sim \text{const.} + |t| \ln |t|$.

We present now the results of the detailed calculation. The $\langle \nu_{\mathbf{q}} \rangle$ of Eq. (22) with Eq. (27) can be evaluated analytically,

$$\begin{aligned} \frac{\langle \nu_{\mathbf{q}} \rangle}{N^{2/3}} &= \frac{1}{N^{2/3}} \sum_{k_2, k_3} \langle n_{\mathbf{k}} \rangle \\ &= -\frac{v^{2/3}}{\lambda^2} \ln [1 - \exp(-x^2 - \tau^2)]. \end{aligned} \quad (36)$$

We used the dimensionless quantities τ of Eq. (31) and $x = q\lambda/\sqrt{4\pi}$. The integral in Eq. (35) becomes

$$\frac{1}{N^{5/3}} \sum_{\mathbf{q}} \langle \nu_{\mathbf{q}} \rangle^2 = \frac{2}{\sqrt{\pi}} \zeta(3/2)^{-5/3} \left(\frac{T}{T_{\lambda}} \right)^{5/2} L(\tau), \quad (37)$$

where $L(0) = 8.30$ and

$$\begin{aligned} L(\tau) &= \int_0^{\infty} dx \left(\ln [1 - \exp(-x^2 - \tau^2)] \right)^2 \\ &= L(0) + 4\pi\tau \ln \tau + 4\pi\tau [\ln(2) - 1] + \mathcal{O}(\tau^2). \end{aligned} \quad (38)$$

The expectation value $\langle \nu_0 \rangle$ is given by Eq. (36) with $x = 0$ because ε_0 has been absorbed in $\tau^2 = -(\mu - \varepsilon_0)/k_B T$ (remark after Eq. (30)). Using $\langle \nu_0 \rangle \sim \ln |t|$, $\langle n_0 \rangle \sim |t|$ and $\tau \sim |t|$ we see that both terms in Eq. (35) yield a contribution $\sim t \cdot \ln |t|$. The critical ($|t| \ll 1$) behaviour of the specific heat reads therefore

$$\begin{aligned} c_V(T, V) &= \left(\frac{\partial(E/N)}{\partial T} \right)_V \\ &= \begin{cases} -A \cdot \ln |t| + B + \dots & (t > 0), \\ -A' \cdot \ln |t| + B' + \dots & (t < 0). \end{cases} \end{aligned} \quad (39)$$

The detailed evaluation of Eq. (35) yields

$$A = A' = 9 \zeta(3/2)^{-2/3} \frac{|w_0|}{v} \frac{1}{N_0^{1/3}} \frac{1}{T_{\lambda}}. \quad (40)$$

The terms depending on the parameter a' cancel. We insert $|w_0|/v$ of Eq. (18) and $T_{\lambda} = 2.17 \text{ K}$ in the result (40).

Experimentally a logarithmic singularity has been measured for c_P over several decades by Ahlers⁸. For saturated vapour pressure we may use $c_V \approx c_P$. (The consistency of a logarithmic form for both, c_V and c_P ,

has been demonstrated by Lee and Puff⁹). The comparison of our result (40) with the experimental values $A \approx A' \approx 0.63 k_B$ (Eq. (48) of Ref. 8) determines the model parameter N_0 ,

$$N_0^{1/3} \approx 50 \quad (\text{adjusted}). \quad (41)$$

This fixes also the small parameter y of Eq. (26) which is a quantitative measure for the deviations of the AIBG from the IBG,

$$y = \frac{|w_0/v| N_0^{-1/3}}{k_B T_\lambda} \approx 0.13. \quad (42)$$

Experimentally one finds a small difference between A and A' (Ahlers⁸ reports 5%) and an indication of a cut of the logarithmic singularity (slightly negative critical exponents⁸ for c_P). In the AIBG a difference between A and A' could be obtained if corrections to approximation (17) are taken into account which differ for $t > 0$ and $t < 0$. A possible cut of the singularity is connected with the lowest possible values of the momenta.

The coefficients B and B' in Eq. (39) can be calculated from Eq. (35), too. They depend on the value of the parameter a' which will be determined only later (Sec. V C). Moreover a generalization of the s.p.f. to be introduced in Sec. IV C will yield another contribution to the jump in $B' - B$.

IV. PHASE ORDERING

We review various aspects of the p.o. assumed in the AIBG. First we show that this p.o. is favoured by the free energy (Sec. IV A). Then we review the necessity of the localization of the s.p.f. for obtaining finite correlation effects due to p.o. (Sec. IV B); this leads to the notion that the coherence range of p.o. approaches infinity at the transition point. Finally (Sec. IV C) we connect the p.o. with a potential superfluid flow.

A. Free energy due to phase ordering

The l.p.o. leads to a lower energy, $E_{\text{corr}}/N \sim (w_0/v) \cdot N_0^{-1/3} < 0$. At the same time such an ordering reduces the entropy, $\Delta S(\text{p.o.}) < 0$. We show that $\Delta S(\text{p.o.})$ is relatively small and that p.o. is therefore favoured by the free energy.

Arbitrary phases in Eq. (2) are statistically equivalent to an equal weight of the two independent choices for the phase, ϕ_0 as in Eq. (3) or $\phi_0 + \pi/2$. We consider p.o. in just one direction. For each \mathbf{k} we have then two s.p. states. The number of possible distributions of n_k bosons on 2 states is $n_k + 1$. The p.o. requires that all n_k bosons go into the same s.p. state; compared to a statistical distribution this implies the entropy change $\Delta s(\text{p.o.}) = -k_B \ln(n_k + 1)$. On the other hand, the n_k

atoms gain by p.o. the energy $\Delta e(\text{p.o.}) \sim (E_{\text{corr}}/N) n_k$. The free energy change Δf is then

$$\Delta f(\text{p.o.}) \sim -\frac{|w_0|}{v N_0^{-1/3}} n_k + k_B T \ln(n_k + 1). \quad (43)$$

In spite of the smallness (42) of the considered correlations the p.o. is favoured ($\Delta f(\text{p.o.}) < 0$) as soon as $n_k \gg 1$. The boson property of the atoms is decisive for this conclusion: It implies that $|\Delta s|$ is not proportional to n_k but only to $\ln n_k$; this behaviour may be paraphrased by ‘bosons like to go into the same state’. For the relevant lowest s.p. states ($k \simeq 0$) the condition $n_k \gg 1$ is fulfilled for $t \ll 1$. Thus p.o. will indeed be adopted when the transition point is approached ($t \rightarrow 0^+$).

As a simplification our ansatz (4) assumes p.o. for all s.p.f. Summing over all momenta and including numerical factors one finds then $E_{\text{corr}} \simeq T \Delta S(\text{p.o.})$ for $T \simeq T_\lambda$ and for the N_0 of Eq. (41). Reversely, the condition $E_{\text{corr}} \sim T_\lambda \Delta S(\text{p.o.})$ constitutes a rough *theoretical* estimate for the model parameter N_0 , leading to an N_0 of the order (41).

B. Phase coherence volume

In the Introduction we argued that for finite correlation effects due to p.o. the s.p.f. have to be localized. We review this point and find that the localization is not required for the lowest s.p. state. We present then an estimate for the extension of this lowest s.p.f.; within this extension the p.o. will be coherent.

The *finiteness* of the correlation energy ($\propto N_0^{-1/3}$) is due to the *finite* spacing $\Delta k = \pi/V_0^{1/3}$ of the q -values in the sum $E_{\text{corr}} \sim \sum \nu_q^2$. Nonlocalized s.p.f. would lead to the same E_{corr} provided that only s.p. states with

$$q_n = q_0 + n \cdot \Delta k, \quad (n = 0, 1, 2, \dots). \quad (44)$$

are occupied. In the macroscopic system the possible q -values are, however, dense, and the entropy drives the particles into the occupation of all available states. This is the reason why a finite Δk can be realized only for s.p.f. localized within a volume $V_0 = (\pi/\Delta k)^3$.

This argument shows that we may either start from finite volumes or from finite Δk , the second feature follows from the first one. There is, however, one difference: Starting from the finite spacing Δk in Eq. (44), the natural condition for the lowest q -value is $q_0 < \Delta k$. The corresponding s.p.f. can then not be localized within V_0 ; as just stated there is also no such need for a localization for the required correlation effect.

In view of this we introduce the following modification of the states (4): Only the s.p.f. with $q_{n>0}$ are localized within V_0 , the s.p.f. with q_0 have a larger extension. We present a crude estimate for the volume V_c of the lowest s.p.f. φ_0 which becomes the condensate state for $t < 0$: Let V_c be some multiple of V_0 ,

that means $V_c = WV_0$. In a volume V_c there are W s.p. states below Δk out of which only one is occupied. A redistribution of the n_0 atoms over these W states would increase the entropy by $\Delta s \simeq k_B \ln(n_0)^W$. At the same time these atoms would loose their correlation energy, $\Delta e \sim n_0 |w_0/v| N_0^{-1/3}$. The stability condition $T_\lambda \Delta s < \Delta e$ yields an upper bound for W . Using $\langle n_0(t > 0) \rangle \sim t^{-2}$ and $\langle n_0(t > 0) \rangle = \mathcal{O}(N)$ we obtain

$$V_c(t) = W V_0 \sim V_0 \cdot \begin{cases} t^{-2} & (t > 0), \\ \infty & (t < 0). \end{cases} \quad (45)$$

This result means the following: Without changing our previous calculations we can modify the states (4) in such a way that the lowest s.p.f. φ_0 has the large and eventually infinite volume V_c , whereas the s.p.f. with $q_n > 0$ are localized. This modification of Eq. (4) has important consequences:

1. Within the volume V_c the direction of p.o. is defined by φ_0 . Therefore, $V_c(t)$ of Eq. (45) is the *phase coherence volume* of the considered p.o.
2. The p.o. constitutes a symmetry breaking because in an infinite system the atoms are free to adopt arbitrary phases ϕ_j in Eq. (2). (The average over the size and shape of finite boxes (assumed in Sec. II A) does not restore this symmetry). The result (45) means that this symmetry breaking changes its character from local to global at the transition point. Approaching the transition point ($t \rightarrow 0^+$) the directions of p.o. of neighbouring V_0 's get aligned, and for $t < 0$ the coherence is potentially infinite.
3. Strictly finite and separated boxes imply a momentum cut at Δk and consequently a cut of the logarithmic singularity. This cut would be described by a lower bound Δk in the integral in Eq. (9). For Eq. (44) we revise the replacement Eq. (9) of the momentum sums by integrals:

$$\left\langle \sum_{q_n=q_0+n \cdot \Delta k} \dots \right\rangle_{q_0} \simeq \frac{1}{\Delta k} \int_{q_{0,\min}}^{\infty} dq \dots \quad (46)$$

The brackets indicate an average over possible q_0 's with $q_0 \leq \Delta k$. The lowest value for q_0 is determined by Eq. (45). A lower bound $q_{0,\min} \sim t^2$ (for $t > 0$) of the integral does not lead to a cut of the logarithmic singularity.

C. Complex phase ordering

The phenomenon of superfluidity can be explained by assuming a macroscopic wave function ψ . The phase $\Phi(\mathbf{r})$ of the complex ψ yields the velocity $v_s = (\hbar/m)\nabla\Phi$ of a potential superfluid flow. In the AIBG the condensed

particles for a macroscopic wave function for $t < 0$, too, Eq. (45). In order to join the common description we replace the real s.p.f. φ_0 by the complex s.p.f.:

$$\bar{\varphi}_0 = \varphi_0 \exp[i\Phi(\mathbf{r})]. \quad (47)$$

This replacement does not affect the previous calculations; moreover, the Jastrow factors in Eq. (5) are without influence on a flow due to $\nabla\Phi \neq 0$. With Eq. (47) all results obtained from the assumption of a macroscopic wave function apply to our model, too. Besides this basic conformity the AIBG leads, however, also to a peculiar modification which will be discussed in the following.

In an IBG-like model (with $\langle n_0 \rangle \sim |t|$ or $\beta = 1/2$) we have to assume that noncondensed particles contribute to the superfluid density ρ_s in order to get agreement with the experiment ($\rho_s \sim |t|^{2/3}$ or $\nu = 1/3$). The considered p.o. should in some way imply that noncondensed particles move coherently with the condensate. We introduce the possibility of a net current of the noncondensed particles by replacing the original real s.p.f. $\varphi_{\mathbf{k}}$ by

$$\bar{\varphi}_{\mathbf{k}} = \varphi_{\mathbf{k}} \exp[i\Phi_{\mathbf{k}}(\mathbf{r})]. \quad (48)$$

A coherent motion with the condensate can now be described by the condition $\Phi_{\mathbf{k}} = \Phi$. We call this condition *complex phase ordering* (c.p.o.) named after the complex phase factors in Eqs. (47) and (48). In contrast to this the p.o. (3) will be called real phase ordering (r.p.o.). We note two points: The spatial correlations of the s.p.f. are unchanged by the replacement (47), (48) because $|\varphi|^2 = |\bar{\varphi}|^2$. Secondly, for a coherent macroscopic flow it is sufficient that the field $\Phi(\mathbf{r})$ is macroscopic; it is then not necessary that *all* contributing s.p.f. are of macroscopic range.

The replacement (47) is suggested by the familiar picture of a superfluid, and Eq. (48) with $\Phi_{\mathbf{k}} = \Phi$ is necessary to reconcile $\beta = 1/2$ (model) with $\nu = 1/3$ (experiment). The fairly straightforward introduction of the phase factors (47) and (48) determines the crucial model prediction for $S_s \neq 0$. The fit to the experimental ρ_s determines the extent of the c.p.o. and consequently S_s .

Presenting a peculiar prediction like $S_s \neq 0$ we feel obliged to make it *quantitative*. For this purpose we have to write down explicit expressions for ρ_s and S_s . This requires additional assumptions which are introduced by plausibility arguments. As explained in the last section, the quantitative result for S_s will not depend sensitively on these assumptions because the model expression for ρ_s is fitted to the experiment.

We require that the new modes $\Phi_{\mathbf{k}}$ do not destroy the correlation energy due to the r.p.o.; this determines the extent of the c.p.o. The field $\Phi(\mathbf{r})$ contains degrees of freedom which will be thermally excited. Let $k_c(t)$ be the average amount of the momenta of this field

$$k_c(t) = \langle |\nabla\Phi(\mathbf{r})| \rangle. \quad (49)$$

Because of $|\varphi|^2 = |\bar{\varphi}|^2$ the fields Φ and $\Phi_{\mathbf{k}}$ do not directly influence the spatial correlations. However, for $\Phi_{\mathbf{k}} \equiv 0$

the noncondensed states \mathbf{k} with $\varepsilon_k < \hbar^2 k_c^2 / 2m$ would have a lower energy than the condensate state. This would invalidate the assumption about the occupation pattern and destroy the correlation energy. On the other hand, for $k \gg k_c$ the $\Phi_{\mathbf{k}}$ will have only minor influence on the occupation pattern. Simplifying and quantifying this qualitative argument leads to the following condition for c.p.o.:

$$\Phi_{\mathbf{k}}(\mathbf{r}) = \begin{cases} \Phi(\mathbf{r}) & (k < k_c), \\ \text{arbitrary} & (k > k_c). \end{cases} \quad (\text{c.p.o.}) \quad (50)$$

This condition means that the localized s.p.f. with $k < k_c$ adopt within their range the (macroscopically defined) phase Φ of the condensate. The physical reason for this c.p.o. is that the correlations due to the r.p.o. (favoured by the free energy) must not be destroyed; in this way the real and complex p.o. are connected to each other.

V. SUPERFLUID DENSITY

A. AIBG expression

Using the new s.p.f. (47) and (48) the states (5) become for $t < 0$:

$$\Psi = F \mathcal{S}(\varphi_0 \exp[i\Phi(\mathbf{r})])^{n_0} \prod_{\mathbf{k}, \text{vol}} (\varphi_{\mathbf{k}} \exp[i\Phi_{\mathbf{k}}(\mathbf{r})])^{n_{\mathbf{k}}}. \quad (51)$$

For these states we evaluate the quantum mechanical expectation value of the current operator. Taking into account the c.p.o. (50) we obtain

$$\begin{aligned} \mathbf{j}(\mathbf{r}, n_{\mathbf{k}}) &= \left\langle \Psi \left| \frac{\hbar}{2i} \sum_{j=1}^N \nabla_j \delta(\mathbf{r} - \mathbf{r}_j) + \text{c.c.} \right| \Psi \right\rangle \quad (52) \\ &= \frac{\rho}{N} \frac{\hbar}{m} \left[(n_0 + \sum_{k < k_c} n_{\mathbf{k}}) \nabla \Phi(\mathbf{r}) + \sum_{k > k_c} n_{\mathbf{k}} \nabla \Phi_{\mathbf{k}}(\mathbf{r}) \right]. \end{aligned}$$

All derivatives of the real functions (F , φ_0 and $\varphi_{\mathbf{k}}$) are cancelled by the complex conjugate (c.c.) term; the only surviving contributions are the phase derivatives. Each such derivative is accompanied by a factor $\langle \Psi | \delta(\mathbf{r} - \mathbf{r}_j) | \Psi \rangle = \rho(\mathbf{r}) / (Nm)$. The mass density ρ is assumed to be a constant in the following.

The contributions of the nonordered phases are averaged out statistically. This is also the case for the thermal excitations of the Φ -field, $\langle \nabla \Phi_{\text{th}} \rangle = 0$. A coherent nonvanishing flow can only be obtained by an additional nonequilibrium contribution Φ_s to Φ ,

$$\Phi = \Phi_{\text{th}} + \Phi_s. \quad (53)$$

The flux due to Φ_s might be relatively stable if it is small enough¹⁰. This implies $|\nabla \Phi_s| \ll k_c$ where $k_c = \langle |\nabla \Phi_{\text{th}}| \rangle$.

Using Eq. (53) and $\langle \nabla \Phi_{\text{th}} \rangle = 0$ we evaluate the statistical expectation value of Eq. (52). Writing $\langle \mathbf{j}_s \rangle = \rho_s \mathbf{v}_s$ with $\mathbf{v}_s = (\hbar/m) \nabla \Phi_s$ we obtain for ρ_s :

$$\frac{\rho_s}{\rho} = \frac{1}{N} \left(\langle n_0 \rangle + \sum_{k < k_c} \langle n_{\mathbf{k}} \rangle \right) = \frac{\rho_0 + \rho_c}{\rho}. \quad (54)$$

This is the AIBG expression for the superfluid density. In a number of points we summarize its implications:

1. The density ρ_s is composed by all atoms adopting the phase $\Phi(\mathbf{r})$. It consists of the condensate density ρ_0 and the comoving density ρ_c . The temperature dependence of this composition (Fig. 1) is obtained by fitting Eq. (54) to the data. The comoving density ρ_c has an internal structure which implies a nonvanishing entropy $S_s \neq 0$ of the superfluid component.

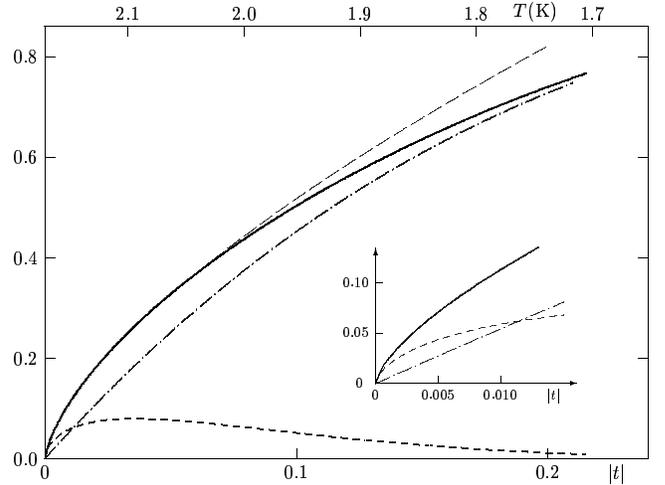


FIG. 1. Composition of the superfluid density ρ_s/ρ (solid line) according to Eq. (54). The ratio ρ_s/ρ is the sum of the model condensate ρ_0/ρ (short dashes) and the coherently comoving density ρ_c/ρ (dash-dotted line). Also shown is the simplest 1-parameter fit $\rho_s/\rho = a_1 |t|^{2/3}$ (long dashes). In the given scale the fitted ρ_s/ρ (solid line) coincides with the data.

2. Expression (54) is the simplest possibility to account in an IBG-like model with $\langle n_0 \rangle \sim |t|$ for $\rho_s \sim |t|^{2/3}$. Fitting Eq. (54) to the experimental ρ_s fixes $k_c(t)$, and consequently the difference between the densities ρ_0 and ρ_s and the model prediction for $S_s \neq 0$.
3. The reproduction of the data with Eq. (54) requires asymptotically $k_c(t) \sim |t|^{2/3}$. Theoretically this asymptotic form is made plausible in Sec. IV B. It leads to a certain functional form of ρ_s which yields excellent fits to the experimental data (Sec. V C).
4. The result (54) contains a specific picture for the relation between the condensate fraction n_c and the

superfluid fraction ρ_s/ρ . The explanation of superfluidity is based on the assumption of a macroscopic wave function. This connection implies, however, an open question: What is the quantitative relation between $n_c(T)$ and $\rho_s(T)/\rho$, and in particular the relation between the $T = 0$ values $n_c \approx 0.1$ (see Ref. 11) and $\rho_s/\rho = 1$? The AIBG proposes the following picture: The model condensate $n_{c,M} = \langle n_0 \rangle / N = \rho_0/\rho$ is the fundamental quantity. It is depleted by the Jastrow factor F in Eq. (51) to the real condensate n_c , for $T = 0$ the value $n_{c,M} = 1$ is reduced to $n_c \approx 0.1$. The factor F in Eq. (51) does, however, not deplete the current due to phase factors. Therefore $n_{c,M} \rightarrow 1$ implies $\rho_s/\rho \rightarrow 1$. On the other hand, for $|t| \ll 1$ the contribution $\rho_c \gg \rho_0$ dominates ρ_s .

B. Effective Ginzburg-Landau model

Ginzburg and Sobyenin¹² have proposed an effective Ginzburg-Landau functional for the free energy F_{GL} as a function of the order parameter field ψ . In this ansatz singular coefficients (like $|t|^{4/3}$ for the $|\psi|^2$ term) are introduced in order to yield the right critical exponents. We follow this kind of approach for investigating the relation between the critical exponents of the order parameter and the superfluid density. This detour is, however, not required for the final result for S_s (see point 2 of Sec. V A).

For discussing the fluctuations of the Φ -field we define the order parameter field by

$$\psi = \sqrt{\frac{n_0}{V}} \exp[i\Phi(\mathbf{r})]. \quad (55)$$

For a definite phase the state (51) must be replaced by the appropriate coherent state¹³. This means that n_0 in Eq. (55) has to be understood as a quantum mechanical expectation value of the occupation number n_0 in such a coherent state.

The statistical expectation value $\langle n_0 \rangle \sim |t|$ can be obtained by minimizing the common Landau energy $F_L/V = R t |\psi|^2 + U |\psi|^4$ (with regular coefficients R and U). Adding a naive kinetic energy term $(\hbar^2/2m)|\nabla\psi|^2$ leads to a violation of scaling invariance and to wrong critical exponents. The decisive feature of the AIBG expression (54) is that the mass density comoving with $\nabla\Phi$ is $\rho_0 + \rho_c$ instead of $\rho_0 = m\langle |\psi|^2 \rangle$ alone. This suggests the following effective Ginzburg-Landau ansatz

$$\frac{F_{GL}}{V} = \frac{\hbar^2}{2m} \frac{\rho_s}{\rho_0} |\nabla\psi|^2 + R t |\psi|^2 + U |\psi|^4. \quad (56)$$

Leaving away unessential constants, the asymptotic behaviour of the mass density ρ_c in Eq. (54) is

$$\rho_c \sim \int_{k < k_c} \frac{d^3k}{k^2 + t^2} \sim k_c. \quad (57)$$

This, $\rho \simeq \rho_c$ for $|t| \rightarrow 0$ and Eq. (49) determine the asymptotic kinetic energy in Eq. (56),

$$\frac{\rho_s}{\rho_0} \langle |\nabla\psi|^2 \rangle = \frac{\rho_c}{m} \langle |\nabla\Phi|^2 \rangle \sim k_c^3. \quad (58)$$

Scaling this kinetic part of F_{GL} with $\langle F_L \rangle \sim t^2$ yields

$$k_c \sim |t|^{2/3}. \quad (59)$$

This implies $\rho_s \sim \rho_c \sim |t|^{2/3}$ and a singular mass coefficient $\rho_s/\rho_0 \sim |t|^{-1/3}$ in Eq. (56). Therefore, Eq. (56) is an effective Ginzburg-Landau ansatz in the same sense as the one by proposed by Ginzburg and Sobyenin¹². Equation (56) demonstrates how the critical exponent $\beta = 1/2$ (from $\langle n_0 \rangle \propto \langle |\psi|^2 \rangle \sim |t|^{2\beta}$) can be connected to $\nu = 1/3$ (from $\rho_s \sim |t|^{2\nu}$) on account of the comoving density ρ_c . The divergent mass coefficient damps the critical fluctuations such that Eq. (56) becomes scaling invariant (this is discussed in more detail in Ref. 14). The scaling invariance implies that Eq. (56) might be used down to $|t| = 0$ and that the critical exponent of k_c and ρ_s might be indeed exactly $2/3$. This possibility is supported by the excellent fits obtained from Eq. (54) with Eq. (59).

C. Fit to the experimental data

We insert the leading $\tau = a'|t|$ and $k_c = k_1|t|^{2/3}$ in Eq. (54). This yields

$$\frac{\rho_s}{\rho} = a_1|t|^{2/3} + a_2|t| + a_3|t|^{4/3} + \dots \quad (\text{MAS}). \quad (60)$$

With 3 parameters we take this as our asymptotic model fit, called MAS. Figure 2 shows that MAS yields an excellent reproduction of the data by Greywall and Ahlers¹⁵ (GA), for saturated vapour pressure.

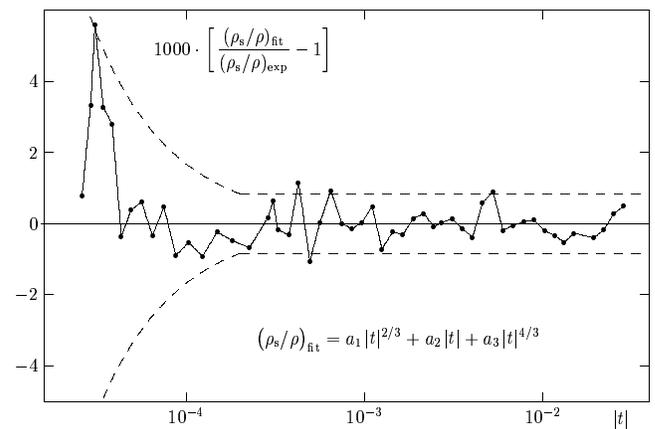


FIG. 2. Asymptotic model fit of the superfluid density. The differences between the 3-parameter fit formula Eq. (60) and the data points are compared to two standard deviations (dashed lines).

The 3-parameter standard fit (SF) used by GA is

$$\frac{\rho_s}{\rho} = a_1 |t|^{a_2} \left(1 + a_3 |t|^{1/2} + \dots\right) \quad (\text{SF}). \quad (61)$$

We compare both fits, MAS and SF, by calculating the sums χ^2 of the quadratic deviations:

$$\frac{\chi_{\text{SF}}^2}{\chi_{\text{MAS}}^2} \simeq 10 \quad (|t| \leq 0.03). \quad (62)$$

This shows that the reproduction shown in Fig. 2 is not a trivial result; in the considered range ($|t| \leq 0.03$) the data cannot be reproduced by SF. We remark that SF (but probably also other 3-parameter fits) fit the data in the considerably smaller range $|t| \leq 0.004$; this range is used by GA for the fit. The range $|t| \leq 0.03$ seems to be appropriate for a 3-parameter fit because already a 1-parameter fit ($a_1 |t|^{2/3}$) roughly reproduces the data in this range (see Fig. 1).

The applicability range of Eq. (54) can be considerably extended by using the following 4-parameter expansion for τ and k_c :

$$\tau(t) = a' |t|, \quad x_c(t) = x_1 |t|^{2/3} + x_2 |t| + x_3 |t|^{4/3}. \quad (63)$$

The form of $x_c = \lambda k_c(t)/\sqrt{4\pi}$ suggests itself because it leaves the expansion (60) unchanged. The 4-parameter model fit, Eq. (54) with Eq. (63), reproduces the data down to 1 K within the experimental errors. The fit yields the parameter values $a' = 3.019$, $x_1 = 2.7028$, $x_2 = -0.837$ and $x_3 = -3.842$; these values have been used for Fig. 1. More details of this fit and a discussion with respect to the quasi-particle picture are given in Ref. 14.

The fit to the experimental data fixes the decomposition of ρ_s into ρ_0 and ρ_c . This decomposition is the starting point for the evaluation of the superfluid entropy S_s .

VI. SUPERFLUID ENTROPY

A. Model prediction

In $\rho_s = \rho_0 + \rho_c$ only the condensate part ρ_0 corresponds to a macroscopic wave function and has thus zero entropy content. The comoving part ρ_c is made up by different s.p.f. (however with the same phase factor in Eq. (48)) and has therefore a nonvanishing entropy content. We determine this entropy.

We start with the well-known entropy expression $S(n_k)$ for a Bose gas with occupation numbers n_k . The IBG equilibrium entropy S_{IBG} is obtained from this expression by $S_{\text{IBG}} = \langle S(n_k) \rangle = S(\langle n_k \rangle_{\text{IBG}})$. In Sec. IV A we have discussed that for r.p.o. in one direction only every second s.p. state is occupied. Since the total particle number $N = \sum n_k$ is fixed we have to put twice as

many atoms in every second state (as compared to the plain IBG expression). This leads to the following model entropy S_M ,

$$S_M(T, V, N) = \frac{k_B}{2} \sum_{\mathbf{k}} \left[(1 + 2\langle n_k \rangle) \ln(1 + 2\langle n_k \rangle) - 2\langle n_k \rangle \ln(2\langle n_k \rangle) \right]. \quad (64)$$

For $t = 0$ this yields $S_M/N = 0.96 k_B$ instead of $S_{\text{IBG}}/N = 1.28 k_B$. The difference $\Delta S(\text{p.o.}) = S_{\text{IBG}} - S_M$ has been compared to $\Delta E(\text{p.o.})$ in Sec. IV A. For $t < 0$ the $\langle n_k \rangle$ with $\tau(t)$ of Eq. (63) will be used in Eq. (64). The resulting overall behaviour of $S_M(T)$ is then—in contrast to $S_{\text{IBG}}(T)$ —similar to that of the experimental entropy $S(T)$.

The entropy of the superfluid component is $S_c = S(\rho_c)$. For calculating S_c we have to restrict the sum in Eq. (64) by $k < k_c$, and to account for the c.p.o. (50). Prescribing the phase field $\Phi_{\mathbf{k}}$ for each s.p.f. in Eq. (48) implies a two to one restriction for each of the three directions of $\nabla \Phi_{\mathbf{k}}$. For each s.p. state \mathbf{k} this reduces the entropy by $k_B \ln 8$ (as long as $n_k \gg 1$). With these specifications we obtain

$$S_c(T, V, N) = \frac{k_B}{2} \sum_{k < k_c} \left[(1 + 2\langle n_k \rangle) \ln(1 + 2\langle n_k \rangle) - 2\langle n_k \rangle \ln(2\langle n_k \rangle) - \ln 8 \right]. \quad (65)$$

The ratio S_c/S_M is the AIBG prediction for the superfluid entropy fraction S_s/S ,

$$\frac{S_s}{S} = \frac{S_c}{S_M}. \quad (66)$$

The calculated result is shown in Fig. 3.

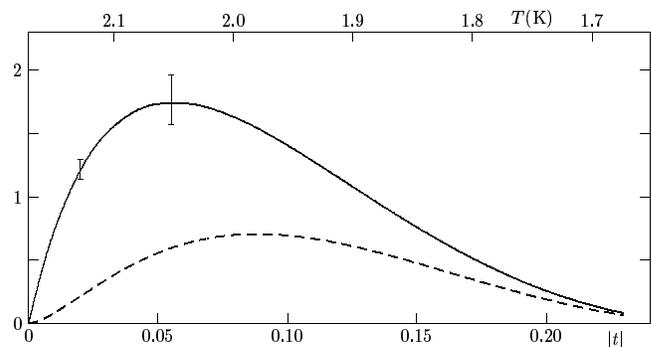


FIG. 3. Model prediction for the superfluid entropy S_s as a function of the temperature. The figure displays the ratios S_s/S (dashed line) and $(S_s/N_s)/(S/N)$ (solid line) both multiplied by a factor 100. The error bars indicate the uncertainty of the fit parameters in Eq. (63). The prediction becomes increasingly uncertain for $|t| > 0.1$ because the calculation is based on an asymptotic expansion (63).

B. Experimental detectability

We show that the AIBG prediction for $S_s \neq 0$ is at the border of present-day experimental detectability. The most prominent experiment showing that $S_s = 0$ (or at least $S_s \approx 0$) is the fountain pressure (FP) measurement. For two containers of liquid helium connected by a superleak a temperature difference dT produces a pressure difference dP . Admitting $S_s \neq 0$ this FP is given by

$$\left(\frac{dP}{dT}\right)_{\text{FP}} = \frac{S}{V} \left(1 - \frac{S_s/N_s}{S/N}\right). \quad (67)$$

For $S_s = 0$ this reduces to the well-known London relation. The correction term is S_s/S multiplied by $N/N_s = \rho/\rho_s$; the model prediction for it is shown in Fig. 3.

The most accurate FP measurements are that by Singaas and Ahlers¹⁶. These authors assume the validity of the London relation and interpret their experiment as an entropy measurement. For this entropy S_{FP} ,

$$\frac{S_{\text{FP}}}{V} = \left(\frac{dP}{dT}\right)_{\text{FP}} \quad (68)$$

they find within the errors no deviation from the true (caloric) entropy S . The absolute values of S near T_λ are, however, uncertain by about 2% (Refs. 16,17).

TABLE I. Comparison between the caloric (S) and the fountain pressure (S_{FP}) entropy. The input is the experimental $S_{\text{FP}}(t)$ and $C_p(t)$; the quantity $\Delta S = S_\lambda - S$ is calculated from C_p . The last column shows the resulting marginal evidence for $S \neq S_{\text{FP}}$ or, equivalently, for $S_s \neq 0$.

1	2	3	4	5
$ t $	$\frac{\rho S_{\text{FP}}}{\text{J cm}^{-3} \text{K}^{-1}}$	$\frac{S_{\text{FP}}}{N k_B}$	$\frac{S_{\text{FP}} + \Delta S}{N k_B}$	$\frac{100(S - S_{\text{FP}})}{S}$
0.0006906	0.2299	0.7571	0.7619 ± 0.0008	0.07 ± 0.11
0.000790	0.2296	0.7561	0.7615 ± 0.0008	0.11 ± 0.11
0.001013	0.2294	0.7555	0.7623 ± 0.0008	0.02 ± 0.11
0.001794	0.2275	0.7493	0.7607 ± 0.0009	0.23 ± 0.11
0.003177	0.2250	0.7411	0.7601 ± 0.0010	0.30 ± 0.13
0.005662	0.2208	0.7274	0.7592 ± 0.0011	0.43 ± 0.14
0.01007	0.2145	0.7069	0.7596 ± 0.0013	0.39 ± 0.17
0.03338	0.1854	0.6116	0.7578 ± 0.0021	0.74 ± 0.34
0.07973	0.1407	0.4648	0.7547 ± 0.0034	1.63 ± 0.71

The theoretical prediction of Fig. 3 suggests that one should compare the temperature dependences of S and S_{FP} near T_λ rather than the absolute values. This is done in Table I in a number of steps:

1. We start with the experimental values¹⁶ for S_{FP} in column 1 and 2. By using $\rho(t)$ from equation (A1) of Ref. 18 we relate the entropy to the particle number rather than to the density (column 3).
2. For column 4 we calculated $\Delta S = S_\lambda - S(T) = \int dT C_P/T$ from the experimental specific heat C_P

of Ref. 8. If S_{FP} were equal to S then column 4 should show the temperature independent $S_\lambda = S(T_\lambda)$.

3. From the temperature dependence of column 4 we deduce $S_\lambda/N = 0.7624 k_B$ as the limit of $S_{\text{FP}} + \Delta S$ for $t \rightarrow 0^-$. Then

$$\frac{S - S_{\text{FP}}}{S} = \frac{S_\lambda - [S_{\text{FP}}(t) + \Delta S(t)]}{S_\lambda - \Delta S(t)} = \frac{S_s/N_s}{S/N}, \quad (69)$$

may be calculated from column 4. The last equality in Eq. (69) follows from Eqs. (67) and (68). Column 5 displays the resulting experimental evidence (or nonevidence) for $S_{\text{FP}} \neq S$.

4. The errors included in the Table are the statistical errors¹⁶ of 0.1% for $S_{\text{FP}}(t)$, and a 1% error for C_P . The systematic errors¹⁶ of $S_{\text{FP}}(t)$ are not included; they should be less important because we consider only the t -dependence (and not the absolute values) in a relatively small interval.

Figure 4 compares column 5 of the Table with our theoretical prediction. There seems to be some indication of a deviation $S \neq S_{\text{FP}}(t)$. The error bars (from the Table) show, however, that the experimental evidence for such a deviation is at most marginally significant. In any case, the compilation of Table I and Fig. 4 shows that, and in which way, our theoretical prediction is within the reach of experimental detectability.

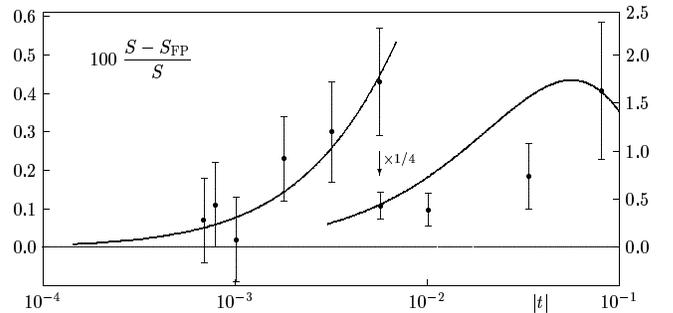


FIG. 4. Difference between the caloric (S) and the fountain pressure (S_{FP}) entropy (dots with error bars, from column 5 of Table 1). This difference is compared to the model prediction (solid line) for $(S_s/N_s)/(S/N)$.

VII. CONCLUDING REMARKS

We have proposed a microscopic model for the λ -transition of liquid helium. The model uses phenomenological assumptions, in particular the transition itself is introduced by the analogy to the IBG. The basic idea of the AIBG is that phase ordering leads to an extra energy $\sim \sum \nu_q^2$ which —evaluated with the IBG occupation numbers— yields a logarithmic singularity. In this

model the superfluid density is not identical to the square of the order parameter field ($\rho_s \neq m|\psi|^2$). A fit to the experimental ρ_s leads to the model prediction for a non-vanishing superfluid entropy S_s .

The model is hardly related to standard models of liquid helium which are either microscopic variational approaches (like Refs. 3,19) or the Landau-Wilson renormalization group theory²⁰. Because of its novelty and originality our model cannot compete in quality of foundation, exactness and completeness with those other theories. It leads, however, to specific predictions about the structure of ρ_s . These predictions are without competition by other approaches and they can and should be tested experimentally. Therefore the presentation of the model seems appropriate in spite of a number of unsolved questions concerning the model assumptions.

In contrast to other approaches the AIBG offers also a possible solution of the so-called microscopic problem of liquid helium. This problem was formulated by Uhlenbeck²¹: If ρ_s is identified with a single quantum state ($\rho_s = m|\psi|^2$) then the approach to equilibrium ($\rho_s = \rho_s(T)$) cannot be understood. In the AIBG this problem is solved by the contribution of noncondensed particles to the superfluid density.

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† This paper was published in Nuovo Cimento **13 D** (1991) 211. It has now been put onto the e-print server because the original source is not readily available. For later work related to the model presented here see cond-mat/9507037, cond-mat/9507038, cond-mat/9901175, cond-mat/0001048 and cond-mat/0106237.

* Electronic address: fliessbach@physik.uni-siegen.de

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